GQMO

Beginner level

Each problem is worth 7 points.

Problem 1

Find all quadruples of real numbers (a, b, c, d) such that the equalities

$$X^2 + aX + b = (X - a)(X - c) \text{ and } X^2 + cX + d = (X - b)(X - d)$$

hold for all real numbers X.

Problem 2

The Bank of Zürich issues coins with an H on one side and a T on the other side. Alice has n of these coins arranged in a line from left to right. She repeatedly performs the following operation: if some coin is showing its H side, Alice chooses a group of consecutive coins (this group must contain at least one coin) and flips all of them; otherwise, all coins show T and Alice stops. For instance, if $n = 3$, Alice may perform the following operations: $THT \to HTH \to HHH \to TTH \to TTT$. She might also choose to perform the operation $THT \to TTT$.

For each initial configuration C, let $m(C)$ be the minimal number of operations that Alice must perform. For example, $m(THT) = 1$ and $m(TTT) = 0$. For every integer $n \geq 1$, determine the largest value of $m(C)$ over all 2^n possible initial configurations C.

Problem 3

Let A and B be two distinct points in the plane. Let M be the midpoint of the segment AB, and let ω be a circle that goes through A and M. Let T be a point on ω such that the line BT is tangent to ω. Let X be a point (other than B) on the line AB such that $TB = TX$, and let Y be the foot of the perpendicular from A onto the line BT.

Prove that the lines AT and XY are parallel.

Problem 4

For all real numbers x, we denote by $\lfloor x \rfloor$ the largest integer that does not exceed x. Find all functions f that are defined on the set of all real numbers, take real values, and satisfy the equality

$$f(x+y) = (-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y)$$

for all real numbers x and y.

Problem 5

Let n and k be positive integers such that $k \leq 2^n$. Banana and Corona are playing the following variant of the guessing game. First, Banana secretly picks an integer x such that $1 \leq x \leq n$. Corona will attempt to determine x by asking some questions, which are described as follows. In each turn, Corona chooses k distinct subsets of $\{1, 2, \ldots, n\}$ and, for each chosen set S, asks the question

"Is x in the set S?"

Banana picks one of these k questions and tells both the question and its answer to Corona, who can then start another turn.

Find all pairs (n, k) such that, regardless of Banana's actions, Corona could determine x in finitely many turns with absolute certainty.

Problem 6

For every integer n not equal to 1 or -1, define $S(n)$ as the smallest integer greater than 1 that divides n. In particular, $S(0) = 2$. We also define $S(1) = S(-1) = 1$.

Let f be a non-constant polynomial with integer coefficients such that $S(f(n)) \leq S(n)$ for every positive integer n. Prove that $f(0) = 0$.

Note: A non-constant polynomial with integer coefficients is a function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k$, where k is a positive integer and $a_0, a_1, \ldots, a_k$ are integers such that $a_k \neq 0$.