GQMO Easy P6 Marking Scheme

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§1 Problem

For every integer n not equal to 1 or -1, define $S(n)$ as the smallest integer greater than 1 that divides n. In particular, $S(0) = 2$. We also define $S(1) = S(-1) = 1$.

Let f be a non-constant polynomial with integer coefficients such that $S(f(n)) \leq S(n)$ for every positive integer n. Prove that $f(0) = 0$.

Note: A non-constant polynomial with integer coefficients is a function of the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k$, where k is a positive integer and $a_0, a_1, \ldots, a_k$ are integers such that $a_k \neq 0$.

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§2 Solutions

Note that for any $n \neq \pm 1$, $S(n)$ must be a prime number. Moreover, by plugging $n = 1$ into the inequality, we deduce that $f(1) = \pm 1$.

We proceed to present two solutions.

Solution 1 (Pitchayut Saengrungkongka)

Suppose that $f(0) \neq 0$. Thus there exists a prime p that does not divide $f(0)$. Consider $n = Mp! + (p-1)! + 1$ where M is a large positive integer.

- First, note that by Wilson's theorem, $n = Mp! + (p-1)! + 1 \equiv 0 + (-1) + 1 = 0 \pmod{p}$ so $p \mid n$. Thus $S(n) \leq p$.
- Now we consider $S(f(n))$. First, note that $f(n) \equiv f(0) \not\equiv 0 \pmod{p}$. Thus $p \nmid f(n)$.
- However, for any prime $q < p$, we find that $f(n) \equiv f(1) = \pm 1 \pmod{q}$. Thus $q \nmid f(n)$.
- Thus if $f(n) \neq \pm 1$, we must have $S(f(n)) > p$, which contradicts to the first point. Hence we are done unless $f(n) = \pm 1$.

Thus we deduce that $f(Mp! + (p-1)! + 1) = \pm 1$ for *any* positive integer M. Hence f must be constant which contradicts the problem's condition.

Remark. Wilson's theorem is not required. We can use Chinese Remainder Theorem to construct n such that $n \equiv 0 \pmod q$ and $n \equiv 1 \pmod{(q-1)!}$.

Solution 2 (Navneel Singhal)

Without loss of generality suppose that $f(1) = 1$ (otherwise replace f with $-f$). Assume for the sake of contradiction that $f(0) \neq 0$, then there exists the *smallest* prime p that does not divide $f(0)$. Now consider an integer $k > 0$ such that

- $p \mid 1 + kf(0)$, and
- $f(1 + kf(0)) \neq \pm 1$.

First, we remark about the existence of k. By inverses modulo p, there exist infinitely many k's that meet the first condition while the second condition rules out only finitely many such k's as those are the roots of $f(1 + xf(0))^2 - 1$, which there are finitely many.

Now we notice the following observations.

- $f(1 + kf(0)) \equiv 1 \pmod{f(0)}$. Since $q \mid f(0)$ for any $q < p$ by the minimality of p, we deduce that $f(1 + kf(0))$ is not divisible by any prime $q < p$.
- Since $p \mid 1 + kf(0)$, we get $f(1 + kf(0)) \equiv f(0) \not\equiv 0 \pmod p$ thus $f(1 + kf(0))$ is also not divisible by p.

As $f(1 + kf(0))$ is not ± 1, we get $S(f(1 + kf(0))) > p$. However, $S(1 + kf(0)) \leq p$ as $p \mid 1 + kf(0)$. This is a contradiction.

§3 Marking Scheme

Comment. To our knowledge, we do not know any completely different approach that solves this problem. If such approach happens, it should be judged as equivalently as possible.

All partial credits are **not addditive**. The correctors are supposed to pick **only one** item that is most favorable to each contestant.

1. Proving **not all** of the following statements. **(0 points)**
   - $S(n)$ is a prime number for any $n \notin \{1, -1\}$.
   - $f(1) \in \{1, -1\}$.
2. Proving that if $f(0) = 0$ and $f(1) = \pm 1$, then the condition $S(f(n)) \leq S(n)$ is true for all positive integer n (i.e. proving the converse of the problem). **(0 points)**

3. Proving **all** of the following statements. **(1 point)**
   - $S(n)$ is a prime number for any $n \notin \{1, -1\}$.
   - $f(1) \in \{1, -1\}$.

4. Proving that $f(0)$ must be even. **(2 points)**

5. Proving that $f(0)$ must be divisible by some **chosen** prime $p \geq 3$. **(3 points)**

6. After the assumption that $f(0) \neq 0$, the contestant has constructed, **without proof**, the correct choice of n that would give the contradiction **if** $f(n) \notin \{1, -1\}$. **(5 points)**

If the contestant had a complete solution, the following deductions might apply. All deductions are **not additive**.

7. If the contestant **omitted the proof** of the fact that $S(n)$ is either 1 or a prime for any integer n, then this omission should be deemed as benign. **(-0 points)**

8. If the contestant directly claimed a contradiction without stating that $f(n)$ could, in principle, have been equal to 1 or -1, then:
   (a) if the contestant provided the construction for **infinitely** many choices of n, then the latter omission should be deemed as benign; **(-0 points)**
   (b) if the contestant provided the construction for **finitely** many choices of n, then the latter omission is not benign, yet very easy to fix. **(-1 points)**

9. The contestant has made other nontrivial minor errors. **(-1 points)**