GQMO Easy P5 Marking Scheme

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§1 Problem

Let n and k be positive integers such that $k \leq 2^n$. Banana and Corona are playing the following variant of the guessing game. First, Banana secretly picks an integer x such that $1 \leq x \leq n$. Corona will attempt to determine x by asking some questions, which are described as follows. In each turn, Corona chooses k distinct subsets of $\{1, 2, \ldots, n\}$ and, for each chosen set S, asks the question

"Is x in the set S?"

Banana picks one of these k questions and tells both the question and its answer to Corona, who can then start another turn.

Find all pairs (n, k) such that, regardless of Banana's actions, Corona could determine x in finitely many turns with absolute certainty.

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§2 Solutions

The answer is Corona wins if and only if $k \leq 2^{n-1}$ or $(n, k) = (1, 2)$. If $n = 1$, then Corona wins without asking any question. From now, assume that $n > 1$. We split the solution into two parts.

Part I: $k > 2^{n-1} \implies$ Banana wins.

There are exactly 2^{n-1} subsets S such that $S \cap \{1, 2\}$ is either $\{1\}$ or $\{2\}$. Thus Banana can always avoid those subsets, thus picking a question that $S \cap \{1, 2\}$ is either $\emptyset$ or $\{1, 2\}$. This way, the output will be exactly the same when $x = 1$ or $x = 2$. Thus Corona will never be able to distinguish between 1 and 2, hence he cannot guess x with certainty.

Part II: $k \leq 2^{n-1} \implies$ Corona wins.

We present two proofs of this part.

Method 1: (Pitchayut Saengrungkongka)

In fact, we will prove that Corona wins in at most n turns. Let T_i denote the set of all possible candidates of x after the i-th turn (thus $T_0 = \{1, 2, \ldots, n\}$).

Claim. Whenever $|T_{i-1}| \geq 2$, Corona could play in the i-th turn so that $|T_i| \leq |T_{i-1}| - 1$.

Proof. We set the i-th questions as follows. Let $r = |T_i|$. Note that there are $2^n - 2^{n-r+1} \geq 2^{n-1}$ subsets S of $\{1, 2, \ldots, n\}$ such that $S \cap T_{i-1}$ is neither the empty set nor T_{i-1}. Thus we can select k of them and form a set of questions. This way, regardless whichever question Banana has answered, some elements of T_{i-1} are declared not to be x. Thus Corona could eliminate at least one number in T_{i-1} hence $|T_i| < |T_{i-1}|$ as claimed. □

Thus Corona can nail down the number of candidates from $n \to n-1 \to \ldots \to 2 \to 1$, eventually guessing the number.

Method 2: (Vincent Jugé)

The idea of this solution is the following claim.

Claim. Suppose that $a, b \in \{1, 2, \ldots, n\}$ where $a \neq b$. Then Corona can ensure that either a or b cannot be the value of x in a single turn.

Proof. There exists exactly 2^{n-1} subsets S such that $S \cap \{a, b\}$ is either $\{a\}$ or $\{b\}$. Hence Corona can pick k of those subsets and form the questions using those subsets. This will ensure that Corona could eliminate either a or b, depending on which is not in the answered S, thus acheiving the claim's objective. □

Back to the main problem. Suppose that Corona apply the claim to all $\binom{n}{2}$ pairs. Since he has applied the claim to the pair (i, x) where $i \neq x$, he has eliminated i for *any* $i \neq x$. Hence he is left with the only possibility for x.

§3 Marking Scheme

Preliminary Remarks:

- This problem should be graded with caution, as many contestants are not very experienced in constructing this type of argument. The graders should pay attention about whether the solution works in any settings, or they have made some invalid assumptions and/or simplifications in order to solve this problem.
- To our knowledge, we expect that most solutions will be similar to the official solutions and we don't know any substantiallly different approach on either part. If such approach happens, it should be judged as equivalently as possible.

Introduction

This problem has two parts, namely

- showing that all pairs (n, k) which $k \leq 2^{n-1}$ work,

- showing that each pair (n, k) which $k > 2^{n-1}$ does not work.

From now on, the word **part** will refer to one of these to parts of the proof.

This problem can be approached in many ways. However, most thought processes will eventually lead to the same solution. We expect the contestant's lines of attack to be in one of the following categories.

(i) Try to optimize the arguments of either part first, then try to prove the other part.

(ii) Ignore the fact that this problem have two parts and try to gain general understandings about all working pairs, which will eventually lead to the characterization of those pairs.

Both (i) and (ii) are reasonable, and eventually lead to the solution. Here we have designed the marking scheme that will allow contestants using (i) and/or (ii) to gain points.

The grading should follow the following procedure.

- If the contestant had the **correct solution** (which might have minor errors), it should be judged as **7 points**, possibly with the 1 point-deduction if there is a nontrivial error.
- **Deduct 1 point** for contestants who had a complete solution, but failed to take the pair $(1, 2)$ into the account.
- If the contestant had an **incomplete solution** or made a critical error on either part. Then there are two ways to evaluate the contestant's progress.
  - **Approach A**, which favors contestants who have made a significant progress on either part. The details are given in **Subsection 3.1**.
  - **Approach B**, which favors contestants who failed to make a significant progress on either part, but have gained some insights that simplifies the problem. The details are given in **Subsection 3.2**.

  The contestant's score should be the **maximum** score based on these two approaches.

§3.1 Approach A

This approach splits into two parts. The contestant's score is the **sum** of the scores from these two parts.

Part 1: Showing that all pairs (n, k) which $k \leq 2^{n-1}$ work.

The full mark of this part is **4 points**, which are divided as follows. All partial credits in this part are **not additive**. In what follows, let T be the set of all possible candidates.

1. Suggest that Corona's strategy is to eliminate numbers apart from x until only x is left. **(1 point)**

2. Prove that if $k \leq 2^n - 2$, then Corona could eliminate at least one possible candidate. **(2 points)**

3. Prove that if $k \leq 2^{n-1}$, then for each a, b, Corona could eliminate either a or b from the set T. **(2 points)**

4. Reduce the problem to showing that for each a, b , Corona could eliminate either a or b in finitely many turns. **(2 points)**

5. Prove that Corona eventually wins **if** he keep giving only sets that neither is disjoint from T, nor contain all elements of T. **(3 points)**

6. Completely solve the either part (through 3+4 or 5 with some more work). **(4 points)**

Part 2: Showing that each pair (n, k) which $k > 2^{n-1}$ does not work.

The full mark of this part is **3 points**, which are divided as follows. All partial credits in this part are **not additive**. In what follows, let y be a number in $\{1, 2, \ldots, n\}$ apart from x.

7. Suggest that Banana's strategy is to keep x and y indistinguible. **(1 point)**

8. Prove that Banana could win **if** he keep picking subsets S which contains both or none of x, y. (i.e. show that this strategy works) **(2 points)**

9. Conclude from 7. that Banana could do that if $k > 2^{n-1}$ **(3 points)**

§3.2 Approach B

Any subset of the following will receive 0 points.

10. Prove that there exists infinitely many tuples that work. **(0 points)**

11. Prove that if (n, k) works, then $(n, k-1)$ and $(n+1, k)$ works. **(0 points)**

12. Try to casework this problem based on small values of k. (such as proving that every n work when $k = 1$.) **(0 points)**

13. Characterize all working pairs for $n \leq 2$. **(0 points)**

14. Prove that $(3, k)$ works for any $k \leq 4$. **(0 points)**

15. Doing other specific cases (such as $(n, 2^n - 1)$ does not work, etc.) that is not clear that whether the contestant understood any general ideas (e.g. through manual calculations). **(0 points)**

The following partial credits can be awarded. All are **not additive**, i.e. the grader is supposed to pick **only one** item that this most favorable for contestants.

16. Correctly guess the answer. **(1 point)**

17. Prove that $(n, 2^{n-1})$ works for a **chosen** $n \geq 4$ **using a generalizable method**. **(2 points)**

18. Prove that $(n, 2^{n-1}+1)$ does not work for a **chosen** $n \geq 3$ **using a generalizable method**. **(2 points)**

19. Prove that Corona wins **if and only if** for every distinct numbers $a, b \in \{1, 2, \ldots, n\}$, he manages to discard at least one of them in finitely many turns. **(2 points)**

20. Prove that Corona wins **if and only if** he can eliminate either 1 or 2 in finitely many turns. **(3 points)**