GQMO Beginner Exam P4 Marking Scheme

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Contents

1. Problem

For all real numbers x, we denote by $\lfloor x \rfloor$ the largest integer that does not exceed x. Find all functions f that are defined on the set of all real numbers, take real values, and satisfy the equality

$$f(x+y) = (-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y)$$

for all real numbers x and y.

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2. Solutions

We claim that $f(x) \equiv 0$ is the only solution to the functional equation. It is simple to check that this function satisfies the functional equation, so we proceed to show that this can be the only solution.

2.1. Solution 1 (Oleg Košik)

Substituting $(1,1)$, we get $f(2) = -2f(1)$.

Substituting $(0.5, 0.5)$, we get $f(1) = 2f(0.5)$.

Substituting $(1, 0.5)$, we get $f(1.5) = f(1) - f(0.5) = f(0.5)$.

Substituting $(1.5, 0.5)$, we get $f(2) = f(1.5) - f(0.5) = 0$.

Using the first and the second equations, we get $f(1) = f(0.5) = 0$.

Substituting $(x, 1)$, we get $f(x+1) = -f(x) + (-1)^{\lfloor x \rfloor} f(1) = -f(x)$.

Substituting $(x, 0.5)$, we get $f(x+0.5) = f(x) + (-1)^{\lfloor x \rfloor} f(0.5) = f(x)$, which implies $f(x+1) = f((x+0.5)+0.5) = f(x+0.5) = f(x)$, which means $f(x) = 0$ for all $x \in \mathbb{R}$.

2.2. Solution 2 (Natanon Therdpraisan)

Plugging $(x, y) = (x, 1)$ into the original equation gives $f(x+1) = -f(x) + (-1)^{\lfloor x \rfloor} f(1)$.

Next, plugging $(x, y+1)$ and $(x+y, 1)$ into the original equation and using the fact above gives

$$\begin{aligned} -f(x+y) + (-1)^{\lfloor x+y \rfloor} f(1) &= f(x+y+1) \\ &= -(-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y+1) \\ &= -f(x+y) + (-1)^{\lfloor x \rfloor + \lfloor y \rfloor} f(1) \end{aligned}$$

Hence, substituting $x = y = 0.5$ gives $f(1) = 0$, for which immediately implies $f(x+1) = -f(x)$.

Plugging $(x, y) = (0, 0)$ into the original equation gives $f(0) = 0$.

Next, plugging in $(x, y+z)$ and (y, z) into the original equation gives $f(x+y+z) = (-1)^{\lfloor y+z \rfloor} f(x) + (-1)^{\lfloor x \rfloor + \lfloor z \rfloor} f(y) + (-1)^{\lfloor x \rfloor + \lfloor y \rfloor} f(z)$.

Therefore, comparing $(x, y+z)$ and $(x+z, y)$ where $0 < x < y < 1$ and $0 < z < 1$ such that $z + x < 1 < z + y$ implies $f(x) = 0 \ \forall \ 0 < x < 1$.

Combining this result with the fact $f(0) = 0$ and $f(x+1) = -f(x)$ thus implies (by induction on $\lfloor x \rfloor$ in both directions) $f(x) = 0$ for all $x \in \mathbb{R}$, whence we are done.

2.3. Solution 3 (Massimiliano Foschi)

Note that, dividing everything by $(-1)^{\lfloor x+y \rfloor}$, we get that $g(x) = \frac{f(x)}{(-1)^{\lfloor x \rfloor}}$ satisfies the Cauchy equation (on $\mathbb{Z}$, where $\lfloor x \rfloor = x$), and thus for integers n, $g(n) = ng(1)$.

Substituting $\left(\frac{1}{2}, \frac{3}{2}\right)$ implies $f(2) = -f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)$ and substituting $\left(\frac{1}{2}, 1\right)$ implies $f\left(\frac{3}{2}\right) = -f\left(\frac{1}{2}\right) + f(1)$.

Then $f(2) = -2f\left(\frac{1}{2}\right) + f(1) = 0$ (the last equality follows from substituting $\left(\frac{1}{2}, \frac{1}{2}\right)$).

As $2g(1) = g(2) = 0$, we must have $g(1) = 0$, so $g(x) = 0$ for every integer x. This gives $f(x) = 0$ for $x \in \mathbb{Z}$ and $f\left(\frac{1}{2}\right) = 0$.

Now choose a such that $\frac{1}{2} < a < 1$. Substituting (a, a) and $(1-a, 1-a)$ yields that $f(2a) = 2f(a)$ and $f(2-2a) = 2f(1-a)$.

Substituting $(2a, 2-2a)$ yields $0 = f(2) = f(2a) - f(2-2a) = 2(f(a) - f(1-a))$, but since $f(a) + f(1-a) = 0$, this yields $f(a) = f(1-a) = 0$, which combined with $f(0) = f\left(\frac{1}{2}\right) = f(1) = 0$ means $f(x) = 0$ for all $0 \le x \le 1$. For every x, we have $f(x) = (-1)^{\lfloor x - \lfloor x \rfloor \rfloor} f(\lfloor x \rfloor) + (-1)^{\lfloor x \rfloor} f(x - \lfloor x \rfloor) = 0$, as required.

2.4. Solution 4 (Navneel Singhal)

Let $P(x, y)$ denote the assertion $f(x+y) = (-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y)$.

$P(x, x)$ gives us that $f(2x) = 2(-1)^{\lfloor x \rfloor} f(x)$.

$P(2x, 2x)$ gives us that $f(4x) = 2(-1)^{\lfloor 2x \rfloor} f(2x) = 4(-1)^{\lfloor 2x \rfloor + \lfloor x \rfloor} f(x)$.

$P(2x, x)$ gives us that $f(3x) = (-1)^{\lfloor x \rfloor} f(2x) + (-1)^{\lfloor 2x \rfloor} f(x) = (2(-1)^{2\lfloor x \rfloor} + (-1)^{\lfloor 2x \rfloor}) f(x)$.

$P(3x, x)$ gives us that $f(4x) = (-1)^{\lfloor x \rfloor} f(3x) + (-1)^{\lfloor 3x \rfloor} f(x) = (2(-1)^{3\lfloor x \rfloor} + (-1)^{\lfloor 2x \rfloor + \lfloor x \rfloor} + (-1)^{\lfloor 3x \rfloor}) f(x)$.

Using the previous expression for $f(4x)$, we get, for all $x \in \mathbb{R}$,

$$(-3(-1)^{\lfloor 2x \rfloor + \lfloor x \rfloor} + 2(-1)^{3\lfloor x \rfloor} + (-1)^{\lfloor 3x \rfloor}) f(x) = 0$$

We break the proof into different cases.

1. The fractional part of x is in $\left[\frac{1}{3}, \frac{1}{2}\right)$, and $\lfloor x \rfloor = n$.

   Then we have $\left(-3(-1)^{2n+n} + 2(-1)^{3n} + (-1)^{3n+1}\right) f(x) = 0$, which gives us that $f(x) = 0$.

2. The fractional part of x is in $\left[\frac{1}{2}, \frac{2}{3}\right)$, and $\lfloor x \rfloor = n$.

   Then we have $\left(-3(-1)^{2n+1+n} + 2(-1)^{3n} + (-1)^{3n+1}\right) f(x) = 0$, which gives us that $f(x) = 0$.

3. The fractional part of x is in $\left[\frac{2}{3}, 1\right)$, and $\lfloor x \rfloor = n$.

   Then we have $\left(-3(-1)^{2n+1+n} + 2(-1)^{3n} + (-1)^{3n+2}\right) f(x) = 0$, which gives us that $f(x) = 0$.

4. The fractional part of x is in $\left(0, \frac{1}{3}\right)$.

   For these x, we shall need a lemma for "lifting the fractional part" of x into one of the above cases.

   Lemma. For any non-negative integer n and real number x,

   $$f(2^n x) = 2^n (-1)^{\sum_{k=0}^{n-1} \lfloor 2^k x \rfloor} f(x)$$

   Proof. The proof shall proceed by induction on n. For $n = 0$, the statement is true. Now suppose we have $f(2^n x) = 2^n (-1)^{\sum_{k=0}^{n-1} \lfloor 2^k x \rfloor} f(x)$ for some n. Then using $P(2^n x, 2^n x)$, we get $f(2^{n+1} x) = 2(-1)^{\lfloor 2^n x \rfloor} f(2^n x) = 2^{n+1} (-1)^{\sum_{k=0}^{n} \lfloor 2^k x \rfloor} f(x)$, which completes the proof of the lemma. □

   Consider the least positive integer n such that $2^n \varepsilon \geq \frac{1}{3}$. Then we claim that the fractional part of $2^n x$ is in $\left[\frac{1}{3}, \frac{2}{3}\right)$. Indeed, if $x = j + \varepsilon$, where $j = \lfloor x \rfloor$, then $2^n x = 2^n j + 2^n \varepsilon$. Suppose $2^n \varepsilon \geq \frac{2}{3}$. Then we would have $2^{n-1} \varepsilon \geq \frac{1}{3}$, which is a contradiction to the minimality of n. And since $2^n \varepsilon$ is in $\left[\frac{1}{3}, \frac{2}{3}\right)$, and because $2^n j$ is an integer, $2^n \varepsilon$ is the fractional part of $2^n x$, and thus the fractional part of $2^n x$ is in $\left[\frac{1}{3}, \frac{2}{3}\right)$. Applying the lemma and the previous part of the proof, we get $0 = f(2^n x) = 2^n (-1)^{\sum_{k=0}^{n-1} \lfloor 2^k x \rfloor} f(x)$, and thus $f(x) = 0$ for all x whose fractional part is in $\left(0, \frac{1}{3}\right)$.

5. The fractional part of x is 0.

   Now we show that for all integers n, we have $f(n) = 0$. In the lemma, set $n = 1$ and $x = \frac{1}{2}$. We get $f(1) = 0$. Now we use induction to show that $f(n) = 0$ for all positive integers n. The base case $n = 1$ has already been established. Now suppose $f(n) = 0$ for some n. Using $P(n, 1)$, we have $f(n+1) = 0 + 0 = 0$, completing the inductive step, and as a result $f(n) = 0$ for all positive integers n.

   $P(0, 0)$ gives us that $f(0) = 0$. Now we use induction to show that $f(-n) = 0$ for all non-negative integers n. The base case $n = 0$ has already been established. Now suppose $f(-n) = 0$ for some n. Using $P(-n-1, 1)$, we have $f(-n) = (-1)^1 f(-n-1) + 0$, which means $f(-n-1) = 0$, completing the inductive step, and as a result $f(-n) = 0$ for all non-negative integers n.

Since each real number has a fractional part in $[0, 1)$, it comes under one of the above 5 cases, and we are done.

2.5. Solution 5 (Hendrik Vija)

Let $P(x, y)$ denote the substitution of x and y into the equation.

$P(x, 0)$ gives us $f(x) = f(x) + (-1)^{\lfloor x \rfloor} f(0) \Longrightarrow f(0) = 0$.

$P(x, -x)$ gives us $0 = f(0) = (-1)^{\lfloor -x \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(-x) \Longrightarrow f(-x) = -(-1)^{\lfloor -x \rfloor - \lfloor x \rfloor} f(x)$.

For $x \notin \mathbb{Z}$, $\lfloor -x \rfloor = -\lceil x \rceil = -\lfloor x \rfloor - 1$, so $f(-x) = -(-1)^{-2\lfloor x \rfloor - 1} f(x) = f(x)$

$P(x, x)$ gives us $f(2x) = 2(-1)^{\lfloor x \rfloor} f(x)$.

$P(2x, -x)$ for $x \notin \mathbb{Z}$ gives

$$\begin{aligned} f(x) &= (-1)^{\lfloor -x \rfloor} f(2x) + (-1)^{\lfloor 2x \rfloor} f(-x) \\ &= 2(-1)^{-\lfloor x \rfloor - 1} (-1)^{\lfloor x \rfloor} f(x) + (-1)^{\lfloor 2x \rfloor} f(x) \\ &= 2(-1)^{-1} f(x) + (-1)^{\lfloor 2x \rfloor} f(x) \\ &= f(x)(-2 + (-1)^{\lfloor 2x \rfloor}) \end{aligned}$$

Thus, either $f(x) = 0$ or $3 = (-1)^{\lfloor 2x \rfloor}$ which is impossible, so $f(x) = 0$ for all $x \notin \mathbb{Z}$.

$P(x, y)$ for $x \notin \mathbb{Z}$ and $y \in \mathbb{Z}$ gives $f(x+y) = (-1)^{\lfloor y \rfloor} f(x) + (-1)^{\lfloor x \rfloor} f(y)$ where using $f(x) = 0$ and $f(x+y) = 0$ (because $x + y \notin \mathbb{Z}$) yields $0 = (-1)^{\lfloor x \rfloor} f(y)$ which in turn yields $f(y) = 0$.

Thus $f(x) = 0$ for all $x \in \mathbb{R}$, which indeed satisfies the equation.

2.6. Solution 6 (Vincent Jugé)

Let $g : x \mapsto |f(x)|$. The function g satisfies the triangle inequality, i.e., $g(x+y) \leq g(x) + g(y)$, with equality if and only if $f(x+y)$, $(-1)^{\lfloor y \rfloor} f(x)$ and $(-1)^{\lfloor x \rfloor} f(y)$ have the same sign.

An immediate induction already shows that $g(nx) \leq ng(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. By setting $x = y$ we also see that $g(2x) = 2g(x)$, and thus that $g(2^k x) = 2^k g(x)$ for all $k \in \mathbb{N}$.

In particular, if $1 \leq a \leq 2^k - 1$ and $b = 2^k - a$, the triangle inequality tells us that

$$2^k g(x) = g(ax + bx) \leq g(ax) + g(bx) \leq (a+b)g(x) \leq 2^k g(x).$$

It follows that $g(ax) = ag(x)$.

This means that $g(ax) = ag(x)$ for all $a \in \mathbb{N}$, and then $f((a+1)x)$ must have the same sign as $(-1)^{\lfloor ax \rfloor} f(x)$ and $(-1)^{\lfloor x \rfloor} f(ax)$. An immediate induction on a then proves that $f((a+1)x)$ has both the sign of $(-1)^{a \lfloor x \rfloor} f(x)$ and $(-1)^{\lfloor ax \rfloor} f(x)$ whenever $a \geq 1$.

Finally, assume that $f(x) \neq 0$ for some $x \neq 0$, and let q be a positive rational such that either $1/2 < x/q < 1$ or $-1/2 < x/q < 0$. Suppose $q = \frac{m}{n}$, where m, n are positive integers. Then we have $mg(x/q) = g(mx/q) = g(nx) = ng(x)$, so $g(x) = qg(x/q)$. Thus, $f(x/q) \neq 0$. Hence, without loss of generality, we assume that $-1/2 < x < 0$ (if we started from a negative x) or $1/2 < x < 1$ (if we started from a positive x), and we know that

$$a \lfloor x \rfloor \equiv \lfloor ax \rfloor \pmod 2$$

for all $a \geq 1$.

However, if $1/2 < x < 1$ and $a = 2$, then $a \lfloor x \rfloor = 0$ and $\lfloor ax \rfloor = 1$. Similarly, if $-1/2 < x < 0$ and $a = 2$, then $a \lfloor x \rfloor = -2$ and $\lfloor ax \rfloor = -1$. Hence, the relation $a \lfloor x \rfloor \equiv \lfloor ax \rfloor \pmod 2$ is invalid in both cases: this proves that $f(x) = 0$ for all $x \neq 0$.

Finally, the original equation states that $f(0) = 2f(0)$, and thus $f(0) = 0$ as well.

3. Preliminary notes on grading this problem

1. This problem needs to be graded with a lot of caution, because we expect a lot of fake-solves due to possible sign errors. The graders are thus requested to carefully examine statements that students claim, and if they come to a correct conclusion with a wrong proof, those proofs should be considered to come into the category of proofs that assume a statement for the sake of completing the proof. For example, if someone shows that $f(1) = 0$ in a wrong way, but then proceeds to complete the solution correctly, they should get points for the rest of the solution, as well as the correct points they made in the proof of $f(1) = 0$.

2. Any complete solutions should be awarded a full 7 points, regardless of whether they fit in the marking scheme or not.

3. Partial credits across different approaches are **not** additive. If a student has a partial solution that can be graded via different marking schemes, the one which leads to the highest score should be followed.

4. Any partial solutions that can lead to solutions, but which are not outlined in one of the following cases, should be graded equivalently.

5. If you are not sure if a partial solution can lead to a solution or not, you could discuss this with other graders and if that turns out to be inconclusive, the members of the problem selection committee.

4. Marking scheme

In every sub-enumeration, the mentioned partials are given, if the mentioned part of the solution has not been completed. For example, if in the first solution, someone has not shown $f(1) = 0$, but has used any three of the four substitutions, then they should get 2 points. The points within the sub-enumerations are not necessarily additive, unless mentioned explicitly.

4.1. Points common to all solutions

1. Only mentioning the correct answer should get 0 points
2. Forgetting to mention that $f(x) = 0$ satisfies the equation should be considered benign, and lead to a deduction of 0 points.
3. Missing cases which can be easily checked (like $f(0) = 0$) should lead to a 1-point deduction.

4.2. Solution 1 (Oleg Košik)

1. Showing that $f(1) = 0$ .... **(3 points)**
   - (a) Using any three of the four substitutions .... **(2 points)**
   - (b) Using any two of the four substitutions .... **(1 point)**
   - (c) Using ≤ 1 of the four substitutions .... **(0 points)**
2. Completing the solution .... **(4 points)**
   - (a) Showing that $f(x + 1) = -f(x)$ .... **(1 point)**
   - (b) Showing that $f(x + 0.5) = f(x)$ .... **(1 point)**
   - (c) Using both of them to show that $f(x) = 0$ .... **(2 points)**
   - (d) Any linear combination of the above .... **(Sum of the corresponding points)**

4.3. Solution 2 (Natanon Therdpraisan)

1. Showing that $f(1) = 0$ .... **(3 points)**
   - (a) Using any three of the four substitutions .... **(2 points)**
   - (b) Using any two of the four substitutions .... **(1 point)**
   - (c) Using ≤ 1 of the four substitutions .... **(0 points)**
2. Showing that $f(x) = 0$ for $0 \leq x < 1$ .... **(3 points)**
   - (a) Only showing $f(0) = 0$ .... **(0 points)**
   - (b) Substituting $(x, y + z)$ or $(x + z, y)$ or both .... **(1 point)**
   - (c) Substituting $(x, y + z)$ and $(x + z, y)$ and comparing the two expressions but not plugging in appropriate values of x, y, z .... **(2 points)**
3. Showing that $f(x) = 0$ for all real x. .... **(1 point)**

4.4. Solution 3 (Massimiliano Foschi)

1. Showing that $f(1) = 0$ .... **(3 points)**
   - (a) Using any three of the four substitutions .... **(2 points)**
   - (b) Using any two of the four substitutions .... **(1 point)**
   - (c) Using ≤ 1 of the four substitutions .... **(0 points)**
2. Showing that $f(x) = 0$ for $0 \leq x < 1$ .... **(3 points)**

(a) Substituting (a,a), $(1-a,1-a)$, $(2a,2-2a)$ for $\frac{1}{2}<a<1$ .... **(3 points)**
(b) Substituting two of them .... **(2 points)**
(c) Substituting one of them .... **(1 point)**

3. Showing that $f(x)=0$ for all real x. .... **(1 point)**

4.5. Solution 4 (Navneel Singhal)

1. Getting to the equation just before the case analysis .... **(2 points)**
2. Solving for the first three cases .... **(1 point)**
3. Solving for the fourth case .... **(2 points)**
   (a) Getting to the lemma by induction .... **(1 point)**
4. Solving for the case of integers .... **(2 points)**
   (a) Showing $f(1)=0$ .... **(1 point)**
   (b) Showing that $f(n)=0$ for either all (non)positive integers or all (non)negative integers .... **(1 point)**

4.6. Solution 5 (Hendrik Vija)

1. Showing $f(x)=0$ for $x\notin\mathbb{Z}$ .... **(5 points)**
   (a) Showing $f(x)=f(-x)$ .... **(2 points)**
      i. Showing that $f(x)=f(-x)$ for a small subset of $\mathbb{R}\backslash\mathbb{Z}$ .... **(1 point)**
   (b) Substituting (x,x) .... **(1 point)**
   (c) Substituting $(2x,-x)$ and concluding $f(x)=0$ for $x\notin\mathbb{Z}$ .... **(2 points)**
   (d) Any linear combination of the above .... **(Sum of the corresponding points)**
2. Showing $f(x)=0$ for $x\in\mathbb{Z}$ .... **(2 points)**

4.7. Solution 6 (Vincent Jugé)

1. Showing that $g(ax)=ag(x)$ for natural a .... **(3 points)**
   (a) Showing that $g(nx)\leq ng(x)$ for $(x,n)\in\mathbb{R}\times\mathbb{N}$. .... **(1 point)**
   (b) Showing that $g(nx)=ng(x)$ for n being a power of 2. .... **(1 point)**
   (c) Using the triangle inequality to show that equality holds. .... **(1 point)**
   (d) Any linear combination of the above .... **(Sum of the corresponding points)**
2. Showing that $f(x)=0$ for all x .... **(4 points)**
   (a) Reducing the case of positive x to $1/2<x<1$ or any other interval that works .... **(1 point)**
   (b) Reducing the case of positive x to $-1/2<x<0$ or any other interval that works .... **(1 point)**
   (c) Choosing a to invalidate the congruence $a\lfloor x\rfloor\equiv\lfloor ax\rfloor \pmod 2$ for positive x .... **(1 point)**
   (d) Choosing a to invalidate the congruence $a\lfloor x\rfloor\equiv\lfloor ax\rfloor \pmod 2$ for negative x .... **(1 point)**
   (e) Any linear combination of the above .... **(Sum of the corresponding points)**