GQMO Easy Problem 3 - Solutions and Marking Scheme

Jakob Jurij Snoj

Because of different possible configurations, angles in all solutions are directed.

Solution 1a (Navneel Singhal):

Let P be the midpoint of BX. Since $\angle TYA = \angle TPA = 90^\circ$, T, Y, P and A are concyclic. Hence, we have

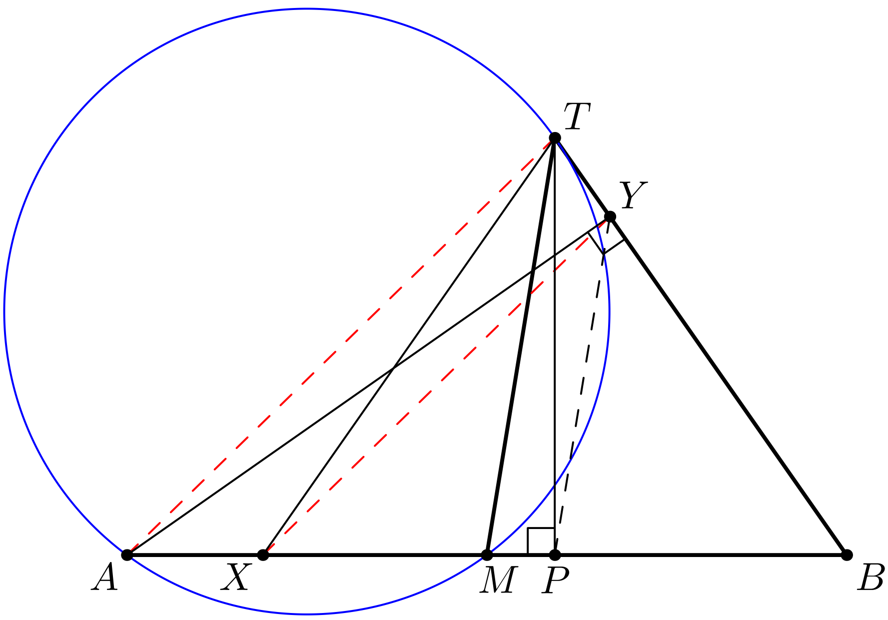
$$\angle MTY + \angle TYP = \angle MAT + \angle TYP = \angle PAT + \angle TYP = 180^\circ,$$

where the first equality holds because BT is tangent to the circumcircle of AMT and the last equality holds because T, Y, P and A are concyclic. It follows that $PY \parallel MT$.

Therefore, we have

$$\frac{BT}{BY} = \frac{BM}{BP} = \frac{BA}{BX},$$

where the last equality holds since M and P are midpoints of BA and BX, respectively. This implies $XY \parallel AT$, as desired.



Solution 1b (Pitchayut Saengrungkongka):

As in Solution 1a, we prove $PY \parallel MT$. Hence, there exists a homothety mapping the segment PY to the segment MT, clearly centered at B. Since $BX = 2BP$ and $BA = 2BM$, we have

$$\frac{BA}{BX} = \frac{BM}{BP},$$

therefore, this homothety also sends X to A. Since it sends Y to T, we establish $XY \parallel AT$, as desired.

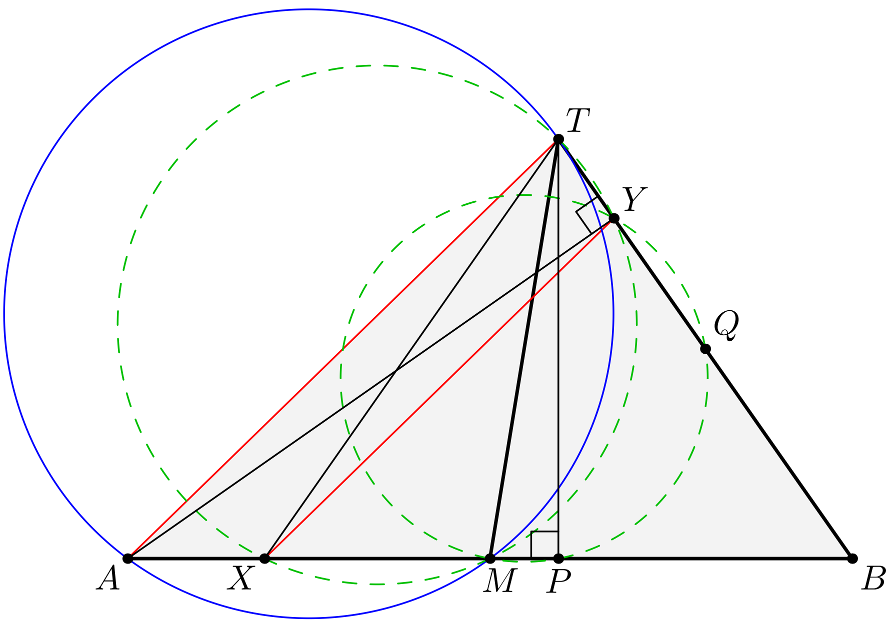
Solution 2a (Jakob Jurij Snoj):

Let P be the midpoint of BX and let Q be the midpoint of BT. Since $TP \perp AB$ and $AY \perp BT$, the points M, P, Q and Y lie on the nine-point circle of the triangle ABT. Hence, by power of a point, $BP \cdot BM = BQ \cdot BY$. Since P and Q are the midpoints of BX and BT, respectively, we also have $BM \cdot BX = BY \cdot BT$, therefore, X, M, Y and T are concyclic.

It now follows that

$$\angle BXY = \angle MXY = \angle MTY = \angle MTB = \angle BAT,$$

where the last equality holds due to the tangency of BT to the circumcircle of AMT. Therefore, $XY \parallel AT$, as desired.



Solution 2b (Jakob Jurij Snoj):

As in Solution 1a, we prove T, Y, P and A are concyclic. It follows that $BP \cdot BA = BY \cdot BT$, which is equivalent to $BX \cdot BM = BY \cdot BT$, since $BA = 2BM$ and $BX = 2BP$. It follows that X, M, Y and T are concyclic. We conclude analogously to Solution 2a.

Solution 3 (Navneel Singhal):

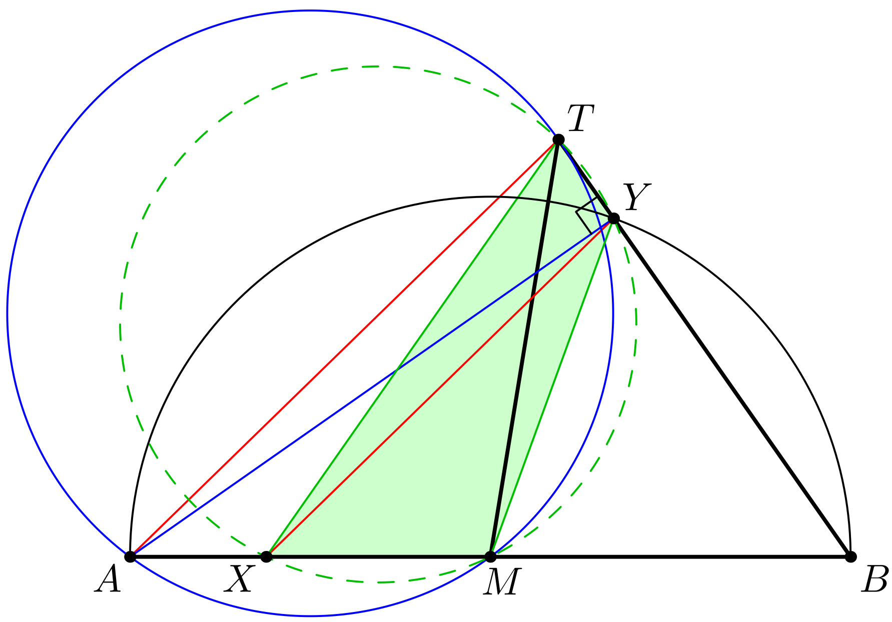
Since $\angle AYB = 90°$, the point Y lies on the circle centered at M going through A and B. Therefore,

$$\angle MYB = \angle TBM = \angle MXT,$$

therefore, M, Y, T and X are concyclic. Hence,

$$\angle MXY = \angle MTY = \angle MAT,$$

where the last equality holds due to the tangency of BT to the circumcircle of AMT. The conclusion follows.



Solution 4 (Michael Reitmeir):

By the intercept theorem $AT \parallel XY$ is equivalent to $\frac{BY}{BT} = \frac{BX}{BA}$. By calculating the power of the point B with respect to the circle ω, we get that $BT^2 = BM \cdot BA$, so after some rearranging we get that it suffices to show that $BY \cdot BT = BM \cdot BX$ which by looking at the power of B again is equivalent to the points $XMYT$ being concyclic.

Notice that since $MA = MB$, we get that ABY lie on a circle with midpoint M by Thales's theorem, which means that $MY = MB$ and therefore $\angle YBM = \angle MYB = 180^\circ - \angle TYM$. Because by definition $TB = TX$, we also know that $\angle BXT = \angle TBX = \angle YBM$, which means that $\angle TXB = \angle TXM = 180^\circ - \angle TYM$. Therefore $XMYT$ are indeed concyclic and we're done.

Solution 5 (Oleg Košik):

By power of point B, we have $BM \cdot BA = BT^2$. (This can be also deduced from the similarity of triangles BAT and MTB.) Since $BA = 2BM$, we get that $BA = BT\sqrt{2}$.

Let $\angle ABT = \alpha$. From triangle ABY we have $BY = BA\cos\alpha$. From isosceles triangle BTX we have $BX = 2BT\cos\alpha$.

Now,

$$\frac{BX}{BY} = \frac{2BT}{BA} = \sqrt{2} = \frac{BA}{BT},$$

which implies $AT \parallel XY$.

Solution 6 (Oleg Košik):

As in Solution 1a, we prove T, Y, P and A are concyclic. It follows that $BP \cdot BA = BY \cdot BT$.

By power of point B with respect to the circumcircle of AMT, we have $BM \cdot BA = BT^2$. (This can be also deduced from similarity of triangles BAT and MTB.)

By dividing the above two equalities, we get

$$\frac{BP}{BM} = \frac{BY}{BT}.$$

Since $BX = 2BP$ and $BA = 2BM$, we get

$$\frac{BX}{BA} = \frac{BY}{BT},$$

which implies $AT \parallel XY$ by intercept theorem.

The points from different solutions are **not** additive. The score of a contestant is the largest possible score on any one solution.

Any papers not following the proposed solutions should be judged as equivalently as possible.

No points are deducted if a contestant fails to account for different possible configurations or exceptional configurations of the problem.

Caveat: Points for computational approaches should only be awarded if the results are interpreted synthetically.

Solution 1:

Points from **1.1**, **1.2**, **1.3** and **1.4** are additive. The point in **1.4 may** be awarded independently of other progress.

1.1) Proving $PY \parallel MT$ .... 4 points
1.2) Observing that $BA/BX = BM/BP$ .... 1 point
1.3) Showing that $BA/BX = BT/BY$ or equivalent .... 1 point
1.4) Showing 1.3 implies $AT \parallel XY$ .... 1 point

If the contestant is awarded no points for **1.1**, the following point may be awarded **instead** (this is **additive** with **1.2**, **1.3** and **1.4**):

1.5.1) Showing A, P, Y, T are concyclic .... 1 point

Solutions 2, 3 and 4:

Points from **2.1** and **2.2** are additive, including the respective partial points. If the contestants fails to prove **2.1** or **2.2**, partial points are given as described below.

2.1) Showing that X, M, Y and T are concyclic .... 4 points

If the contestant does not prove **2.1**, partial progress is rewarded **instead**. The following points are **not additive**:

2.1.1) Showing A, P, Y and T are concyclic .... 1 point
2.1.2) Showing M, P, Q, Y are concyclic .... 1 point
2.1.3) Showing 2.1.1 or 2.1.2 along with all length relations required to deduce 2.1 by power of a point .... 2 points
2.1.4) Showing A, B and Y lie on a circle centered in M .... 1 point

2.2) Reducing the problem to X, M, Y and T being concyclic .... 3 points

If the contestant does not prove **2.2**, partial progress is rewarded **instead**:

2.2.1) Showing $BT^2 = BM \cdot BA$ and interpreting the required condition as $BY/BT = BX/BA$ . . . . 1 point

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Solution 5:

Points from **5.1**, **5.2** and **5.3**, including the respective partial points, are additive. The point in **5.3 may** be awarded independently of other progress.

5.1) Expressing, with proof, BX/BY in terms of BA and BT only (such as $BX/BY = 2BT/BA$) . . . . 3 points

If the contestant does not prove **5.1**, partial progress is rewarded **instead**:

5.1.1) Noting at least one of the equalities $BY = BA\cos\alpha$ and $BX = 2BT\cos\alpha$ or equivalent . . . . 1 point

5.2) Concluding $BX/BY = BA/BT$ from 5.1 . . . . 3 points

If the contestant does not prove **5.2**, partial progress is rewarded **instead**:

5.2.1) Proving $BA = BT\sqrt{2}$ . . . . 2 points

5.3) Concluding $AT \parallel XY$ from the obtained length relations . . . . 1 point

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Solution 6:

Points from **6.1**, **6.2** and **6.3**, including the respective partial points, are additive. The point in **6.3 may** be awarded independently of other progress.

6.1) Proving $BP/BM = BY/BT$ (or any length relation that implies $BX/BA = BY/BT$ together with $BX = 2BP$, $BA = 2BM$ and $BT = 2BQ$) . . . . 5 points

If the contestant does not prove **6.1** but establishes any length relations that are together sufficient for obtaining **6.1** only with algebraic manipulation, such as both $BP \cdot BA = BY \cdot BT$ and $BM \cdot BA = BT^2$, **3 points** are awarded for **6.1**.

If the contestant does not prove **6.1** and does not establish the aforementioned length relations, partial points can be awarded. The following points are **additive**:

6.1.1) Showing A, P, Y and T are concyclic or BTP and BAY are similar triangles or equivalent (such as proving both equalities in 5.1.1) . . . . 1 point
6.1.2) Proving $BM \cdot BA = BT^2$ . . . . 1 point

6.2) Concluding $BX/BA = BY/BT$ or equivalent .... 1 point
6.3) Showing 6.2 implies $AT \parallel XY$ .... 1 point

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