Exam: beginners - P2 Markscheme

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8 May 2020

1 The problem

The Bank of Zürich issues coins with an H on one side and a T on the other side. Alice has n of these coins arranged in a line from left to right. She repeatedly performs the following operation: if some coin is showing its H side, Alice chooses a group of consecutive coins (this group must contain at least one coin) and flips all of them; otherwise, all coins show T and Alice stops. For instance, if $n = 3$, Alice may perform the following operations: $THT \to HTH \to HHH \to TTH \to TTT$. She might also choose to perform the operation $THT \to TTT$.

For each initial configuration C, let $m(C)$ be the minimal number of operations that Alice must perform. For example, $m(THT) = 1$ and $m(TTT) = 0$. For every integer $n \geq 1$, determine the largest value of $m(C)$ over all 2^n possible initial configurations C.

2 Solutions

What follows is an outline of some of the solutions to this problem. The scope of this section is to make graders acquainted with the approaches students are more likely to use while attempting to solve this problem. These are not meant to be full solutions and shall not be used as a model.

Let $f(n)$ be the maximum of $m(C)$ over all n-coin configurations.

2.1 Upper bound

2.1.1 Approach 1

It can be noted that $m(C)$ is $\leq$ the number of groups of consecutive H's in C (in every move, Alice can flip each one of them). Then, it can be proven that the number of consecutive H's is $\leq \lceil \frac{n}{2} \rceil$. Thus, $m(C) \leq \lceil \frac{n}{2} \rceil$ for every C.

2.1.2 Approach 2

Note that $m(C)$ is finite. Note that Alice's moves are commutative. Note that, if two groups Alice flips share the same endpoint (that is, the space between at the right of the right-most coin or at the left of the left-most coin), then she could have flipped their union (or intersection), thereby using one move less.

Consider a sequence of moves Alice could use: if it consists of more than $\lceil \frac{n}{2} \rceil$ moves, then at least two groups share an endpoint and that sequence of moves is not optimal.

2.1.3 Approach 3

If there are at most $\lceil \frac{n}{2} \rceil$ H's, Alice can flip them one by one, using at most $\lceil \frac{n}{2} \rceil$ moves. If there are more than $\lceil \frac{n}{2} \rceil$ H's, then there are at most $\lfloor \frac{n}{2} \rfloor - 1$ T's. Alice can flip them one by one and then flip again all the coins, thus using at most $\lfloor \frac{n}{2} \rfloor$ moves.

2.2 Lower bound

Both approaches focus on the configuration $HTHT...$

2.2.1 Approach 1

Let $a_0 a_1 a_2 \ldots a_n a_{n+1}$ be a binary string such that, for every $i \in \{1, 2, \ldots, n\}$, $a_i = 1$ if the i-th coin shows H and 0 otherwise and $a_0 = a_{n+1} = 0$. Let $k(C)$ be the number of sequences $(a_i, a_{i+1}, \ldots, a_j)$ such that $1 \leq i \leq j \leq n$, $a_i = a_{i+1} = \ldots = a_j = 1$ and $a_{i-1} = a_{j+1} = 0$.

For every $i \in \{0, 1, \ldots, n\}$, let i be a "wrong index" if $a_i \neq a_{i+1}$. Clearly, there are $2k(C)$ wrong indices. After the flip of some coins $a_i, \ldots, a_j$ the only indices which can change there status are $i - 1$ and j. Thus, the number of indices decreases by at most two with every move and at least $k(C)$ indices are needed.

Then simply note that $k(HTHT...) = \lceil \frac{n}{2} \rceil$.

This solution can be phrased differently by using the $(n+1)$-uple $(a_1 - a_0, a_2 - a_1, \ldots, a_{n+1} - a_n) \in \mathbb{Z}_2^{n+1}$ but the key idea is identical.

2.2.2 Approach 2

Consider an optimal sequence of moves. If two sequences consist of the same number of moves, but in one of them the number of flips of the single coin is lower, than, for the sake of this proof, that sequence shall be considered optimal. Suppose that there exists two groups A and B, flipped by Alice, such that $A \cap B \neq \emptyset$. Then, if $B \subseteq A$, we note that Alice could have flipped the one or two groups that make up $A \setminus B$. If $A \subseteq B$ Alice can do an analogous thing. Otherwise, Alice could have flipped $A \setminus B$ and $B \setminus A$.

In all cases, this algorithm provides Alice with a better sequence of moves. Thus, as at least one optimal sequence exists, the groups of coins flipped by Alice in that optimal sequence are disjoint: in other words, every coin is flipped at most once.

It's easy, now, to conclude that Alice needs at least $\lceil \frac{n}{2} \rceil$ moves.

3 Markscheme

Seeing as this is the second problem in the exam for beginners, this markscheme has been designed in order to reward intuition, even in those cases where the students has not been rigorous. Therefore, graders are invited to examine contestants' solutions carefully, in order to discern correct intuitions from rough work.

There are **3 points** given for the upper bound and **4 points** given for the lower bound. Scores obtained from those two bounds are mostly independent and completely additive.

The only observations that are worth points are those mentioned in the markscheme, possibly put differently. If the grader believes that an observation is not mentioned in the markscheme but could be worth points then they should discuss the issue publicly.

Throughout the markscheme, a function $g(n) : \mathbb{Z} \to \mathbb{R}_{>0}$ is said to be good if $\mid g(n) - \lceil \frac{n}{2} \rceil \mid \leq 1$ for every integer n.

3.1 Non-rewarded observations

Any subset of this observations is worth **0 points**:

- $f(n)$ is finite
- $f(n) \geq g(n)$ or $f(n) \leq g(n)$ for some bad function $g(n)$
- Alice's moves are commutative
- Alice should not make the same move twice
- If there is a move that transforms C into C', then $\mid m(C) - m(C') \mid \leq 1$
- Any estimate or claim on the average value of $m(C)$
- If there is a move that transforms C into C', then there is a move that transforms C' into C
- Discussion of particular configurations that do not make it clear that the student has understood any general things

3.2 Upper bound - 3 points

The following marks are, in general, not additive:

- **1 point**: The contestant has understood that $m(C)$ is maximum when H's and T's alternate, has provided an algorithm that works for at least two of the $n \geq 6$ cases (but has not necessarily described it in general), has shown or outlined how it works for those configurations C (s)he believes achieve the maximum and has claimed that the answer is "about $\frac{n}{2}$", "asymptotically $\frac{n}{2}$" or "$g(n)$", where $g(n)$ is a good function. This point shall not be given in case there is more than one non-discarded claim and at least one of them is not a good function. This point shall still be given if the student does not mention an explicit function but there is no doubt (s)he has understood what the maximal configuration is like and how to go from it to $TT...TT$. Therefore, if the student does not show that (s)he believes that is the maximum (even by writing "max" next to the example), this point shall not be given. As a general rule of thumb, this point shall be given only if the students proves to have understood some general regularities but is not be able to write them down. That is, no points shall be given for mere rough work.

- **2 points**: The contestant satisfies the criteria needed to get 1 point and has come up with the right claim, possibly put in a different way (for example, $\frac{n}{2}$ for n even and $\frac{n+1}{2}$ for n odd).
- **2 points**: The contestant has a full proof for the upper bound but miscalculates the final answer, however (s)he still writes a good function.
- **2 points**: The contestant has a full proof for the upper bound that follows approach 1 but does not prove that consecutive groups of H's are at most $\lceil \frac{n}{2} \rceil$.
- **2 points**: The contestant has a full proof for the upper bound that follows approach 3 but miscalculates something (for example, the number of H's and/or T's) and writes down false inequalities.
- **3 points**: The contestant has a full flawless proof for the upper bound.

In addition to this, **1 point** shall be given if the contestant proves that if Alice flips two groups which share an endpoint, then that sequence of moves is not optimal (or stronger claims), as long as (s)he has understood that Alice's moves are commutative. This point shall only be given if the students would have 0 or 1 point otherwise (for the upper bound). That is, 0+1=1, 1+1=2, 2+1=2.

3.3 Lower bound - 4 points

The following marks are, in general, non-additive:

- **1 point**: Proof that if $A \cap B \neq \emptyset$, then A and B can be replaced as in 2.2.2.
- **2 points**: Claim that the number of blocks consisting of consecutive H's can decrease by at most one. This point shall be given if the contestant claims an equivalent thing, for example that the number of k such that $a_k \neq a_{k+1}$ (see 2.2.1) decreases by at most two, as long as (s)he considers placeholders a_0 and a_{n+1} (or adds $a_1 + a_n$).
- **3 points**: Proof that the number of blocks consisting of consecutive H's can decrease by at most one. This point shall be given if the contestant proves an equivalent thing, for example that the number of k such that $a_k \neq a_{k+1}$ (see 2.2.1) decreases by at most two, as long as (s)he considers placeholders a_0 and a_{n+1} (or adds $a_1 + a_n$).
- **3 points**: Any complete proof that uses approach 2 but misses a trivial case (for example, when A and B share an edgepoint).
- **3 points**: Any complete proof by a student who miscalculates the answer and writes a good function. These points shall not be given if the mistake is not due to a miscalculation but, for example, the student's proof provides a weaker bound than $\lceil \frac{n}{2} \rceil$.
- **4 points**: Any full flawless proof for the lower bound.