Find all quadruples of real numbers (a, b, c, d) such that the equalities

$$X^2 + aX + b = (X - a)(X - c) \text{ and } X^2 + cX + d = (X - b)(X - d)$$

hold for all real numbers X.

Answer The only solutions are $(0, 0, 0, 0)$ and $(-1, -2, 2, 0)$.

Solution 1 (Morteza Saghafian)
This solution uses Vieta's formulas for polynomials. In this case, Vieta's formulas imply

$$\begin{cases} a &= -(a + c) \\ b &= ac \\ c &= -(b + d) \\ d &= bd \end{cases}$$

The problem comes down to solving this system of equations. The first equality implies $c = -2a$. So, the second equality reads $b = ac = -2a^2$ and the third equality implies $d = -b - c = 2a^2 + 2a$. We can rewrite the fourth equality as

$$2a(a + 1)(2a^2 + 1) = 0.$$

This implies $a = 0$ or $a = -1$. In the first case we obtain $(a, b, c, d) = (0, 0, 0, 0)$ and in the second $(a, b, c, d) = (-1, -2, 2, 0)$.

Since we are considering all Vieta's formulas at once, any solution of the system is automatically a solution of the initial problem. So, we conclude that $(a, b, c, d) = (0, 0, 0, 0)$ and $(a, b, c, d) = (-1, -2, 2, 0)$ are indeed solutions. (It is of course also possible to plug in the two solutions of the system in the initial equations to check that they are indeed solutions.)

Remark The Vieta formulas can be re-obtained by expanding the products on the right hand side of the two equations and stating that equality holds for all X if and only if the coefficients in front of the terms of the same degree agree.

Solution 2 (Vincent Jugé)
In this solution, we simply plug in some values for X, since both equality must hold for every real number X. With $X = a$ in the first equality we obtain $2a^2 + b = 0$ and with $X = c$ we obtain $c^2 + ca + b = 0$. With $X = b$ in the second equality we obtain $b^2 + cb + d = 0$ and with $X = d$ we obtain $d^2 + cd + d = 0$. So, the problem comes down to solving the following system of equation

$$\begin{cases} 2a^2 + b &= 0 \\ c^2 + ca + b &= 0 \\ b^2 + cb + d &= 0 \\ d^2 + cd + d &= 0 \end{cases}$$

Similarly to the previous solution, this system has the following solutions: $(a, b, c, d) = (0, 0, 0, 0)$ and $(a, b, c, d) = (-1, -2, 2, 0)$.

With this approach, an actual check is needed to verify that $(a, b, c, d) = (0, 0, 0, 0)$ and $(a, b, c, d) = (-1, -2, 2, 0)$ are indeed solutions of the initial problem. In the fist case, we obtain twice $X^2 = X^2$ and in the second $X^2 - X - 2 = (X + 1)(X - 2)$ and $X^2 + 2X = (X + 2)X$. All these equations hold for every real number X and so we are done.

Remark It is possible to make different substitutions to get another system of equations that leads to the desired solutions (eg: $X = 0$ and $X = 1$ in both equalities).

Marking scheme The problem has two main steps: Finding a system of equations whose solutions are the solutions of the problem and solving that system.

a) Partial solutions are graded as follows:

The first point is awarded for writing down the complete set of solutions and the remaining six points for checking that these are indeed the unique solutions.

this should be (-1,-2,2,0)

- 1P: claiming that $(0,0,0,0)$ and $(1,-2,2,0)$ are the only solutions of the initial problem, or of an equivalent system of equations, or of any equivalent formulation of the initial problem. No explicit check is required.
- 4P: obtaining a system of equations in a, b, c, d whose solutions are the solutions of the problem. Partial credits are distributed as follows:
  - +1P: for each of the four equations needed in a system that leads to the correct solution. For instance by plugging in $X = a, b, c$ or d, or $X = 0, 1$, or by applying Vieta's formulas to one of the two equations only (one application of Vieta leading to two equations is of course worth two points).
- 2P: solving the system completely. Partial credits are distributed as follows:
  - +1P: expressing three of the four variables in terms of the fourth, or reducing to the case $b = 1$ or $d = 0$.

b) Possible deduction:

The following deduction applies only if 4P or more were attributed in part a).

- -1P: if the students does not mention explicitly that the solutions he obtained by solving some system actually make the initial equations hold for every real number X