GQMO Advanced Exam Marking Scheme

GQMO PSC

Contents

0. Problems

Problem 1

Let ABC be a triangle with incentre I. The incircle of the triangle ABC touches the sides AC and AB at points E and F, respectively. Let ℓ_B and ℓ_C be the tangents to the circumcircle of BIC at B and C, respectively. Show that there is a circle tangent to EF, ℓ_B and ℓ_C with centre on the line BC.

Proposed by Navneel Singhal, India

Problem 2

Geoff has an infinite stock of sweets, which come in n flavours. He arbitrarily distributes some of the sweets amongst n children (a child can get sweets of any subset of all flavours, including the empty set). Call a distribution of sweets k-*nice* if every group of k children together has sweets in at least k flavours. Find all subsets S of $\{1, 2, \dots, n\}$ such that if a distribution of sweets is s-nice for all $s \in S$, then it is s-nice for all $s \in \{1, 2, \dots, n\}$.

Proposed by Kyle Hess, USA

Problem 3

We call a set of integers *special* if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \dots, 4n\}$ cannot be partitioned into n disjoint special sets.

Proposed by Mohsen Jamali, Iran

Problem 4

Prove that for all sufficiently large integers n, there exist n numbers $a_1, a_2, \dots, a_n$ satisfying the following three conditions:

1. Each number a_i is equal to either -1, 0 or 1.
2. At least $2n/5$ of the numbers $a_1, a_2, \dots, a_n$ are non-zero.
3. $a_1/1 + a_2/2 + \dots + a_n/n = 0$.

Proposed by Navneel Singhal, India, Kyle Hess, USA, and Vincent Jugé, France

Problem 5

Let $\mathbb{Q}$ denote the set of rational numbers. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that, for all $x, y \in \mathbb{Q}$,

$$f(x)f(y+1) = f(xf(y)) + f(x).$$

Proposed by Nicolás López Funes and José Luis Narbona Valiente, Spain

Problem 6

Decide whether there exist infinitely many triples (a, b, c) of positive integers such that all prime factors of $a! + b! + c!$ are smaller than 2020.

Proposed by Pitchayut Saengrungkongka, Thailand

Problem 7

Each integer in $1, 2, 3, \ldots, 2020$ is coloured in such a way that, for all positive integers a and b such that $a+b \leq 2020$, the numbers a, b and $a+b$ are not coloured with three different colours. Determine the maximum number of colours that can be used.

Proposed by Massimiliano Foschi, Italy

Problem 8

Let ABC be an acute scalene triangle, with the feet of A, B, C onto BC, CA, AB being D, E, F respectively. Suppose N is the nine-point centre of DEF, and W is a point inside ABC whose reflections over BC, CA, AB are W_a, W_b, W_c respectively. If N and I are the circumcentre and incentre of $W_aW_bW_c$ respectively, then prove that WI is parallel to the Euler line of ABC.

Note: If XYZ is a triangle with circumcentre O and orthocentre H, then the line OH is called the Euler line of XYZ and the midpoint of OH is called the nine-point centre of XYZ.

Proposed by Navneel Singhal, India and Massimiliano Foschi, Italy

1. Problem 1

1.1. Problem

Let ABC be a triangle with incentre I. The incircle of the triangle ABC touches the sides AC and AB at points E and F, respectively. Let ℓ_B and ℓ_C be the tangents to the circumcircle of BIC at B and C, respectively. Show that there is a circle tangent to EF, ℓ_B and ℓ_C with centre on the line BC.

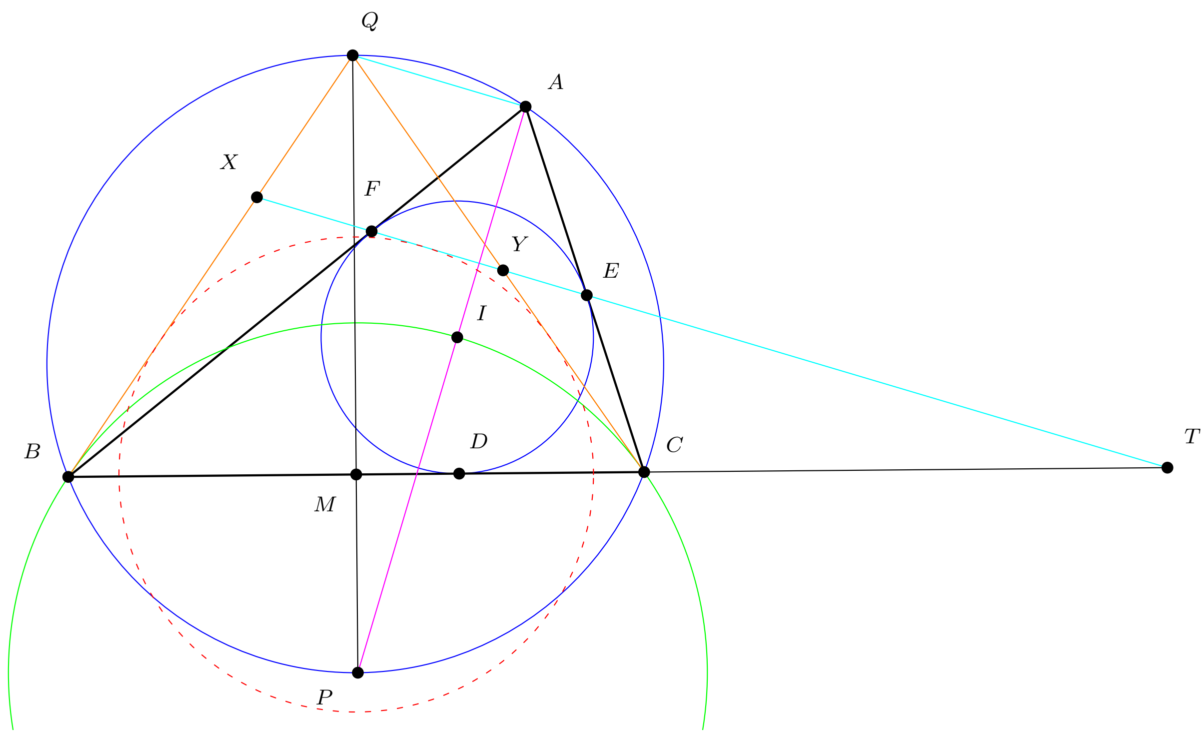
Proposed by Navneel Singhal, India

1.2. Common definitions and results

Let M be the midpoint of BC and D the remaining intouch point (opposite A). Let P be the intersection of the angle bisector of $\angle BAC$ and the circumcircle of ABC. Let Q be the intersection of the tangents to the circumcircle of BIC in B and C. Let X, Y and T be the intersections of EF with ℓ_B, ℓ_C and BC, respectively.

It is well known that P is the circumcentre of BIC (by the Incentre-Excentre Lemma[1]). Since QB and QC are tangent to the circumcircle of BIC, we have $\angle PBQ = \angle QCP = 90°$, which implies P, Q, B and C are concyclic, therefore Q also lies on the circumcircle of ABC. Additionally, by Thales' Theorem, $\angle QAP = 90°$ and therefore, $QA \parallel EF$.

Since $\angle XQY = \angle BAC$ and $\angle YXQ = 180° - \angle XQA = \angle ACB$, the triangles ABC and QYX are similar.



[1] Let ABC be a triangle, I its incentre and I_A its excentre opposite A. The points B, C, I and I_A lie on a circle centred in the midpoint of the arc BC not containing A on the circumcircle of ABC. The proof is by relatively straighforward angle chasing.

1.3. Solutions

1.3.1. Solution 1 (Fedir Yudin)

We claim M is the centre of the desired circle. To prove this, we will show the distances of M to ℓ_B, ℓ_C and EF are equal.

Since M is the midpoint of BC, we have

$$d(M, EF) = \frac{d(B, EF) + d(C, EF)}{2}.$$

Let $\alpha = \angle BAI = \angle IAC = \frac{1}{2}\angle BAC$. By considering the lines parallel to AI going through B and C, we can see

$$\frac{d(B, EF) + d(C, EF)}{2} = \frac{BF \cdot \cos\alpha + CE \cdot \cos\alpha}{2} = \frac{(BD + DC)\cos\alpha}{2} = MB\cos\alpha,$$

where the last equality holds since $BF = BD$ and $DC = EC$.

Since QB is tangent to the circumcircle of BIC, we have

$$\angle BQM = 90^\circ - \angle CBQ = 90^\circ - (180^\circ - \angle BIC) = 90^\circ - \angle CBI - \angle ICB = \alpha.$$

We conclude by noting that MBQ and the triangle formed by BM, ℓ_B and the perpendicular from M to ℓ_B are similar, which implies

$$MB\cos\alpha = d(M, \ell_B).$$

Due to symmetry, it is clear that $d(M, \ell_B) = d(M, \ell_C)$. Therefore, by combining the above relations, we obtain

$$d(M, EF) = d(M, \ell_B) = d(M, \ell_C).$$

Thus, M is indeed the centre of the desired circle.

1.3.2. Solution 2 (Navneel Singhal, Jakob Jurij Snoj)

Let ℓ be the line tangent to the circumcircle of BIC in I, which is clearly parallel to EF and AQ. Since $\angle PQC = \angle PAC = \angle IAE$ and $\angle AEI = 90^\circ = \angle QCP$ due to PQ being the diameter of the circumcircle of ABC, the triangles AEI and QCP are similar.

Since AEI and QCP are similar triangles and BC and EF their altitudes, we can now compute:

$$\frac{d(Q, EF)}{d(Q, \ell)} = \frac{d(A, EF)}{d(A, \ell)} = \frac{QM}{QP}.$$

Observe the homothety centred in Q taking P to M. As it preserves tangency, it sends the circumcircle of BIC to another circle ω, tangent to ℓ_B and ℓ_C, centred in M. Additionally, due to the above equality, it sends ℓ to EF. Since ℓ is tangent to the circumcircle of BIC, EF is tangent to ω. It follows that ω is the desired circle.

1.3.3. Solution 3 (Massimiliano Foschi, Navneel Singhal)

Note B, C, E and X are concyclic since $\angle BXE = 180^\circ - \angle ACB$.

Observe that, since AD, BE and CF are concurrent, the harmonic property of complete quadrilaterals implies T and D are harmonic conjugates with respect to A and B. In particular, we have $TD \cdot TM = TB \cdot TC$, since TD and TM are harmonic and arithmetic means, respectively, of TB and TC.

By taking the power of point T with respect to the circumcircle of $BCEX$, we now obtain

$$TE \cdot TX = TB \cdot TC = TM \cdot TD,$$

which implies E, X, M and D are concyclic.

We can now obtain

$$\angle MXE = \angle CDE = \frac{180^\circ - \angle ACB}{2} = \frac{\angle BXE}{2},$$

which implies XM is the angle bisector of $\angle BXY$. Analogously, we obtain YM is the angle bisector of $\angle XYC$. This implies M is the centre of the Q-excircle of QXY, which concludes the proof.

1.3.4. Solution 4 (Pitchayut Saengrungkongka)

Since AQ is parallel to EF, we have

$$\frac{FX}{QA} = \frac{BF}{BA} \qquad \text{and} \qquad \frac{EY}{QA} = \frac{CE}{CA}.$$

By Menelaus' Theorem for triangle AEF and line BC, we have

$$\frac{AC}{CE} \cdot \frac{ET}{TF} \cdot \frac{FB}{BA} = 1,$$

which, along with the above equalities, implies

$$\frac{TE}{TF} = \frac{EY}{FX}.$$

Let M' be the excentre of the Q-excircle of QXY (the desired circle). Since

$$\angle M'XY = \frac{\angle BXY}{2} = \frac{\angle BQA}{2} = \angle CDE = \angle DFE,$$

XM' is parallel to FD. We conclude by noting the homothety centred in T which takes E to Y evidently also takes F to X and D to M'. It follows that M' indeed lies on BC as desired.

1.3.5. Solution 5 (Pitchayut Saengrungkongka)

Since AQ is parallel to EF, we have

$$\angle XFB = \angle QAB = \angle QCB = \angle TBX,$$

therefore, BXT and FXB are similar triangles. Similarly, we can prove $\angle YEC = \angle TCY$, therefore TCY and YEC are similar triangles. It follows

$$TX = TB \cdot \frac{BX}{FB} \quad \text{and} \quad FX = FB \cdot \frac{XB}{BT}, \quad \text{therefore} \quad \frac{TX}{FX} = \frac{TB^2}{BF^2}.$$

Similarly, we obtain $TY/EY = TC^2/CE^2$. We will now prove these two quantities are equal.

As in Solution 3, we prove $(B, C; D, T) = -1$. It follows that

$$\frac{TX}{FX} = \frac{TB^2}{BF^2} = \frac{TB^2}{BD^2} = \frac{TC^2}{CD^2} = \frac{TC^2}{CE^2} = \frac{TY}{EY}.$$

We conclude analogously to Solution 4 by defining M' and observing the homothety from T taking X to F and Y to E also takes M' to D, therefore, M' lies on BC as desired.

1.3.6. Solution 6 (Pitchayut Saengrungkongka)

As in the other solutions, we prove $(B, C; D, T) = -1$. It follows that

$$MD \cdot MT = MB \cdot MC = MP \cdot MQ,$$

which implies MDP and MTQ are similar triangles (in particular, this shows D is the orthocentre of PQT).

Let I_A be the centre of the A-excircle of ABC. It is well known that the incircle and excircle tangency points on a triangle side are symmetric with respect to the side's midpoint. A homothety centred in A sends D to the point diametrically opposite the excircle tangency point on BC on the A-excircle of ABC, which implies this point is collinear with A and D. Together with the above, this implies MI_A and AD are parallel by observing a homothety with ratio $1/2$ centred in the tangency point of the A-excircle and BC.

The triangles DIA and MPI_A therefore have pairwise parallel sides and are thus similar. In particular, let G be the intersection of BC and AI. The point G is the centre of the homothety taking DIA to MP_IA, which implies

$$\frac{DG}{GM} = \frac{AG}{GI_A}.$$

Recall once again PMD and TMQ are similar triangles, as well as $\angle HTM = \angle APH$. This implies $DG/GM = QH/HM$ and therefore,

$$\frac{QH}{HM} = \frac{AG}{GI_A}.$$

We conclude by noting the triangles ABC and QYX are similar. Indeed,

$$\angle ACB = 180^\circ - \angle BQA = 180^\circ - \angle BXY = \angle YXQ \quad \text{and} \quad \angle XQY = \angle BAC.$$

Since AI and QM are respective angle bisectors in these triangles, the equality $QH/HM = AG/GI_A$ implies that, since I_A is the excenter opposite A of ABC, M is the excenter opposite Q of QXY. The Q-excircle of QXY is precisely the desired circle, which concludes the proof.

1.3.7. Solution 7 (Pitchayut Saengrungkongka)

We use barycentric coordinates with respect to $\triangle ABC$, i.e. point (x, y, z) where $x + y + z = 1$ will refer to $x\vec{A} + y\vec{B} + z\vec{C}$. We also use the notation $(kx : ky : kz)$ to denote point (x, y, z) for any $k \in \mathbb{R}$.

Set $a = BC, b = CA, c = AB$. It's easy to see that $I = (a : b : c)$, $E = (s - c : 0 : s - a)$ and $F = (s - b : s - a : 0)$. Now we proceed to compute Q. Since Q lies on the A-external bisector, it follows that $Q = (t : -b : c)$ for some $t \in \mathbb{R}$. As in Solution 2, we get that Q lies on the circumcircle of ABC. Thus by plugging in this into the circumcircle equation, we get

$$a^2(-b)(c) + b^2(t)(c) + c^2(-b)(t) = 0 \implies t = \frac{a^2}{c - b}$$

Thus Q has coordinate $(-a^2 : b(b - c) : c(c - b))$. Now to compute X, one could intersect BS with EF directly. But this requires factorization of asymmetric expression, so we will present a simpler approach: let X' be the point on BS such that $MX' \parallel DF$ and we aim to show that $X' \in EF$.

To compute X', note that since $X' \in BS$, there exists $t \in \mathbb{R}$ such that $X' = (-a^2 : t : c(c - b))$. Clearly, M has the coordinate $(0 : 1 : 1)$ while ∞_{DF} has the coordinate $(a : c - a : -c)$ (as it's also the point at infinity along the B-external bisector). By Shoelace formula, we get that

$$0 = \det \begin{bmatrix} 0 & 1 & 1 \\ a & c - a & -c \\ -a^2 & t & c(c - b) \end{bmatrix} = a \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & c - a & -c \\ -a & t & c(c - b) \end{bmatrix}$$

Expanding via minors (on the first row), we find that

$$\det\begin{bmatrix}1 & -c\\ -a & c(c-b)\end{bmatrix} = \det\begin{bmatrix}1 & c-a\\ -a & t\end{bmatrix} \implies t + a(c-a) = c(c-b) - ac$$

Thus $t = a^2 + c^2 - bc - 2ac$. Therefore $X' = (-a^2 : a^2 + c^2 - bc - 2ac : c(c-b))$. To show that this point lies on EF, we need to show that

$$\det\begin{bmatrix}-a^2 & a^2+c^2-bc-2ac & c^2-bc\\ s-c & 0 & s-a\\ s-b & s-a & 0\end{bmatrix} = 0$$

Expanding the determinant, we have to show that

$$(a^2 + c^2 - bc - 2ac)(s-a)(s-b) + c(c-b)(s-c)(s-a) = -a^2(s-a)^2.$$

Dividing by $s - a$, and multiplying both sides by 2, it suffices to show that

$$(a^2 + c^2 - bc - 2ac)(a + c - b) + c(c - b)(a + b - c) + a^2(b + c - a) = 0.$$

The first term is

$$\begin{aligned}(a^2 + c^2 - bc - 2ac)(a + c - b) &= (a^3 + ac^2 - abc - 2a^2c) + (a^2c + c^3 - bc^2 - 2ac^2)\\ &\quad - (a^2b + c^2b - b^2c - 2abc)\\ &= a^3 - ac^2 + abc - a^2c + c^3 - 2bc^2 - a^2b + b^2c\end{aligned}$$

while the last two terms combined is

$$\begin{aligned}&c(ac + bc - c^2 - ab - b^2 + bc) + a^2(b + c - a)\\ =\ &ac^2 + 2bc^2 - c^3 - abc - b^2c + a^2b + a^2c - a^3\end{aligned}$$

It's clear that all add up to 0 so we get that $X' \in EF$ so $X' = X$. This completes the problem as we have $MX \parallel DF$ thus $\angle MXE = \angle DFE = 90^\circ - \frac{C}{2}$ and $\angle MXY = \angle AQX = \angle C$ hence MX bisects $\angle BXE$.

1.3.8. Solution 8a (Navneel Singhal)

We first prove another stronger property of the configuration: the circumcircle of QXY is tangent to the circumcircle of BIC.

Note that, by the Right Angles on Intouch Chord Lemma, we need to show the following:

Let ABC be a triangle, and let $A'B'C'$ be its orthic triangle, and H its orthocenter. Let the tangents to the circumcircle of BHC at B and C and the line $B'C'$ form a triangle GUV, with G being on both tangents, and V being on that from C. Then the circumcircle of GUV is tangent to the circumcircle of BHC.

This can be done using Casey's theorem, however we follow a more synthetic approach.

We claim that the point of tangency is K, the A-Humpty point. Let U' be the point where $B'C'$ meets the circumcircle of $BFKM$ (where M is the midpoint of BC). Then we have $\angle U'BK = \angle U'C'K = \angle B'C'K = \angle B'AK = \angle KCB = \angle UBK$, so $U = U'$.

Also $\angle UKB = \angle UKC' + \angle C'KB = \angle UBC' + \angle C'MB = \angle B - \angle A + 2\angle C'CB = \angle C = \angle B'CK + \angle KCB = \angle UVK + \angle KCB$, so the circumcircle of UKV is tangent to the circumcircle of BKC, which is also the circumcircle of BHC.

Now by Miquel's theorem on triangle GBC and points U, V, M, we have K on the circumcircle of GUV too, so we are done.

We now conclude by noting that $\angle BUM = \angle BC'M = \angle MBC' = \angle MUE$ (which does not need the full power of the result), or by simply using the properties of the mixtilinear excircle (the midpoint of the touchchord of the mixtilinear excircle is the corresponding excentre).

1.3.9. Solution 8b (Pitchayut Saengrungkongka)

Let E_1 and F_1 be the intersections of BI and CI, respectively, with EF. Note that, by the Right Angles on Intouch Chord Lemma, we have $MB = MC = ME_1 = MF_1$.

It follows that

$$\angle CME_1 = 180^\circ - \angle E_1MB = 2\angle MBI = \angle MBA = \angle CYE,$$

therefore, the points C, M, Y and E_1 are concyclic. We can now conclude by noting

$$\angle XYM = \angle E_1CM = \angle ME_1C = \angle MYC,$$

from which it follows that YM is the angle bisector of XYC. By showing analogously that XM is the angle bisector of BXY or remarking QM is the angle bisector of BQC, it follows that M is indeed the desired excentre.

1.3.10. Solution 8c (Navneel Singhal)

We define E_1 and F_1 as in Solution 8b. Since $AQ \parallel EF$, it follows by Reim's Theorem that B, F, Y and C are concyclic. By the extended Right Angles on Intouch Chord Lemma, E_1 lies on the line connecting the midpoints of BC and CA. Since this line is parallel to BF, it follows by Reim's Theorem that M, C, E_1 and Y are concyclic. We now finish as in Solution 8b.

1.3.11. Solution 9 (Pitchayut Saengrungkongka)

Let H be the orthocenter of BIC, K be the midpoint of BH and B_1, C_1 be the feet of perpendiculars from C to HB and B to HC. From the Right Angles on Intouch Chord Lemma, B_1, C_1, E, F are colinear.

As $DI \perp BC$, the points K, M, D, B_1 and C_1 all lie on the nine-point circle of BCH. We now apply Pascal's theorem on $KMDB_1C_1K$ to get that $KM \cap EF$, $MD \cap C_1K$ and $KK \cap DB_1$ are collinear. Note that, since D, I, B_1 and C are concyclic,

$$\angle CDB_1 = \angle CIB_1 = 180^\circ - \angle BIC = \angle CBI + \angle ICB = 90^\circ - \angle BAI 90^\circ - \angle BQP = \angle CBQ,$$

so $BQ \parallel DB_1$. Now, since $KD = KB_1$ due to Thales' Theorem, K is the midpoint of arc DC_1B_1 in the nine-point circle of BIC. It follows the tangent at K to it is parallel to DB_1. From the result we obtained by using Pascal's Theorem, it now follows that $KM \cap EF = X$.

Since

$$\angle C_1B_1I = \angle C_1HI = \angle C_1HD = \angle C_1CD = \angle ICD = \angle IB_1D,$$

HC is one of the angle bisectors of EF and DB_1. Since $DB_1 \parallel BQ$ and $CH \parallel MK$ and since $X = KM \cap EF$ lies on BQ, it follows that XM is also an angle bisector of $\angle BXY$. It follows that M is the desired excentre, which concludes the proof.

1.3.12. Solution 10 (Amirhosein Rajabi)

Without loss of generality, we can assume $\angle B \geq \angle C$. One can see that AQ is parallel to EF and we have that since $\angle AEX = \angle QBC$, $XECB$ is a cyclic quadrilateral. Let the bisectors of the angles between EF and BQ intersect BC at L and N (with L between B and T). Therefore, $(T, B; L, N) = -1$. Also it is known that $(T, D; B, C) = -1$. Now we use the similarity of triangles TXB and TCE (because of the cyclic quadrilateral $XECB$) along with these harmonic ratios to obtain

$$\frac{BT}{BD} = \frac{TC}{CD} = \frac{TC}{CE} = \frac{TX}{XB} = \frac{TL}{LB} = \frac{NT}{BN} = \frac{NB + BT}{DN + BD} = \frac{NB}{DN}$$

This means $\frac{NT}{BN} = \frac{NB}{DN}$, which gives us $NB^2 = ND \cdot NT$. So the reflection of B over N is the point C' such that $(B, C'; D, T) = -1$, and since $(B, C; D, T) = -1$, we have $C = C'$, so N is the midpoint of BC, and we are done.

1.4. Preliminary notes on grading this problem

1. Any complete solutions should be awarded a full 7 points, regardless of whether they fit in the marking scheme or not.
2. Partial credits across different approaches are **not** additive. If a student has a partial solution that can be graded via different marking schemes, the one which leads to the highest score should be followed.
3. Any partial solutions that can lead to solutions, but which are not outlined in one of the following cases, should be graded equivalently.
4. If you are not sure if a partial solution can lead to a solution or not, you could discuss this with other graders and if that turns out to be inconclusive, the members of the problem selection committee.
5. Non-trivial but minor errors in a complete solution usually lead to a deduction of 1 point.
6. No points are deducted if a contestant fails to handle possible configuration issues (such as not using directed angles).
7. No points are deducted for quoting well known results without proof (including, but not limited to, Right Angles on Intouch Chord Lemma, Incentre-Excentre Lemma and the harmonic property of complete quadrilaterals).
8. **Caveat:** Points for computational approaches should only be awarded if the results are interpreted synthetically.

1.5. Marking scheme

In every sub-enumeration, the mentioned partials are given, if the mentioned part of the solution has not been completed. For example, if in the first solution, someone has not expressed $d(M, EF)$ as described, but has proven $d(M, EF) = (d(B, EF) + d(C, EF))/2$, the contestant would get 1 point for that part of the solution. The points within the sub-enumerations are **not** additive, unless mentioned explicitly.

1.5.1. Solution 1 (Fedir Yudin)

1. Expressing $d(M, l_B)$ or $d(M, l_C)$ only in terms of MB, MC or BC and α (such as $MB\cos\alpha$) .... **(1 point)**
2. Expressing $d(M, EF)$ only in terms of MB, MC or BC and α (such as $MB\cos\alpha$) .... **(5 points)**
   (a) Proving $d(M, EF) = (d(B, EF) + d(C, EF))/2$ .... **(1 point)**
   (b) Proving $d(M, EF) = ((BF + CE)\cos\alpha)/2$ .... **(3 points)**
3. Argumenting $d(M, \ell_B) = d(M, \ell_C) = d(M, EF)$, which implies M is the desired centre (this point can only be given if 1. and 2. are both proven) .... **(1 point)**

1.5.2. Solution 2 (Navneel Singhal, Jakob Jurij Snoj)

1. Proving $AQ \parallel EF$ **and** proving Q lies on the circumcircle of ABC .... **(1 point)**
2. Reducing the problem to proving $d(Q, EF)/d(Q, \ell) = QM/QP$ .... **(2 points)**
   (a) Only mentioning a homothety centred in Q which sends the circumcircle of BIC to the desired circle without making the above reduction .... **(1 point)**
3. Proving $d(Q, EF)/d(Q, \ell) = QM/QP$ .... **(4 points)**
   The following partial points are **additive**:
   (a) Proving $d(Q, EF)/d(Q, \ell) = d(A, EF)/d(A, \ell)$ .... **(1 point)**
   (b) Proving AEI and QCP are similar triangles .... **(1 point)**

1.5.3. Solution 3 (Massimiliano Foschi, Navneel Singhal)

1. Proving B, C, E, X are concyclic or proving B, C, F, Y are concyclic .... **(2 points)**
   (a) Proving $AQ \parallel EF$ **and** proving Q lies on the circumcircle of ABC .... **(1 point)**
2. Proving M, D, E, X are concyclic or proving M, D, F, Y are concyclic .... **(4 points)**
   (a) Proving $(B, C; D, T) = 1$ .... **(1 point)**
3. Reducing the problem to one of the concyclicities in 2. .... **(1 point)**

1.5.4. Solutions 4 and 5 (Pitchayut Saengrungkongka)

1. Reducing the problem to showing the homothety sending EY to FX is centred at T (or an equivalent statement) .... **(2 points)**
   (a) Only defining the desired excentre M' **and** proving $XM' \parallel FD$ or $YM' \parallel ED$, but not explicitly reducing the problem .... **(1 point)**
2. Proving the homothety sending EY to FX is centred at T (or an equivalent statement) .... **(5 points)**
   (a) Proving $FX/QA = BF/BA$ **or** $EY/QA = CE/CA$ **or** BXT and FXB are similar triangles **or** TCY and YEC are similar triangles .... **(2 points)**
   (b) Only using the Menelaus' theorem for triangle AEF and line BC or proving an equivalent statement which would, together with (a), imply 2. .... **(2 points)**
   (c) Only showing $(B, C; D, T) = -1$ .... **(1 point)**
   (d) Proving $TX/FX = TB^2/BF^2$ or $TY/EY = TC^2/CE^2$ .... **(3 points)**
   (e) Proving length relations that would, only with algebraic manipulation, be sufficient to establish 2., but failing to complete the proof .... **(4 points)**
   (f) Proving $AQ \parallel EF$ (this point is **additive only with (b) and (c)**) .... **(1 point)**

1.5.5. Solution 6 (Pitchayut Saengrungkongka)

1. Proving M, Q, T, D form an orthocentric system .... **(2 points)**
   (a) Only showing $(B, C; D, T) = -1$ .... **(1 point)**
2. Proving $DG/GM = AG/GI_A$ .... **(2 points)**
   (a) Proving $MI_A \parallel AD$ .... **(1 point)**
3. Reducing the problem to $DG/GM = AG/GI_A$ .... **(3 points)**
   (a) Only reducing the problem to $QH/HM = AG/GI_A$, regardless of whether 1. was completed **(2 points)**
   (b) Only proving ABC and QYX are similar or an equivalent statement .... **(1 point)**
   (c) Only proving $DG/GM = QH/HM$ .... **(1 point)**
   (d) Proving (b) and (c), but failing to complete the proof .... **(2 points)**

1.5.6. Solutions 8 and 9 (Navneel Singhal, Pitchayut Saengrungkongka)

1. Defining E_1 and F_1 such as in Solution 8b or equivalent **and** using the Right Angles on Intouch Chord Lemma to prove their properties, thus transforming the problem as in one of these solutions (for example, defining E_1 and F_1 in Solution 8b and proving $MB = MC = ME_1 = MF_1$ or $BE_1 \perp E_1C$ in the same notation, such as in Solution 9) .... **(2 points)**

2. Solving the transformed problem .... **(5 points)**

   (a) Only proving ABC and QYX are similar, B, C, Y, F or B, C, E, X are concyclic or an equivalent statement .... **(1 point)**

   (b) Proving M, C, E_1, Y or M, B, F_1, X (in notation of 8b) are concyclic .... **(3 points)**

   (c) Using Pascal's Theorem such as in Solution 9 in a way that allows the contestant to finish in an analogous manner .... **(2 points)**

   (d) Proving M, K and X are collinear .... **(3 points)**

1.5.7. Solution 10 (Amirhosein Rajabi)

1. Proving X, E, C and B are concyclic or proving Y, F, C and B are concyclic .... **(1 point)**

2. Obtaining $NT/BN = NB/DN$ or equivalent, where N is defined as the intersection of the internal bisector of $\angle BXY$ or $\angle XYC$ and line BC .... **(5 points)**

   (a) Observing L and N and establishing $(T, B; L, N) = -1$ **and** $(T, D; B, C) = -1$ .... **(2 points)**

   (b) Proving only one of the two harmonic ratios .... **(1 point)**

3. Reducing the problem to $NT/BN = NB/DN$, where N is defined as in 2. .... **(1 point)**

2. Problem 2

2.1. Problem

Geoff has an infinite stock of sweets, which come in n flavours. He arbitrarily distributes some of the sweets amongst n children (a child can get sweets of any subset of all flavours, including the empty set). Call a distribution of sweets *k-nice* if every group of k children together has sweets in at least k flavours. Find all subsets S of $\{1, 2, \ldots, n\}$ such that if a distribution of sweets is s-nice for all $s \in S$, then it is s-nice for all $s \in \{1, 2, \ldots, n\}$.

Proposed by Kyle Hess, USA

2.2. Solutions

We claim that $S = \{1, 2, \ldots, n\}$ is the only S that satisfies the problem conditions. It is clear that it does. Now we show that this is the only one.

Consider the obvious translation to the language of bipartite graphs, where one part represents children and the second one represents flavours, and two vertices are connected if and only if the respective child has a sweet of the respective flavour. We show that for each $r \in \{1, 2, \ldots, n\}$, there exists a bipartite graph (with two n-sized partitions) that is not r-nice but is s-nice for all $\in \{1, 2, \ldots, n\} \setminus \{r\}$. Then for every $S \neq \{1, 2, \ldots, n\}$ our construction for some $r \in \{1, 2, \ldots, n\} \setminus S$ proves that S doesn't satisfy problem conditions.

2.2.1. Solution 1 (David Rusch)

We take the bipartite graph $\{1, \ldots, r\} \times \{1, \ldots, r-1\} \cup \{r+1, \ldots, n\} \times \{1, \ldots, n\}$. Note that the sets are defined to be empty if they don't make sense ($r = 1$ or $r = n$). Now, r doesn't work because of $\{1, 2, \ldots, r\}$, but every other s works obviously. If $s < r$, there are at least $r - 1$ neighbors no matter what and if $s > r$, there are n neighbours.

2.2.2. Solution 2 (Navneel Singhal)

We construct such bipartite graphs by induction. The case for $\{1, 2\}$ is easy to see.

Now suppose we have such examples for n. We wish to exhibit such examples for $n + 1$. We break into two cases: $r \neq n + 1$ and $r = n + 1$.

Firstly suppose $r \neq n+1$. For this case, join $n+1$ on the left to all vertices on the right, then utilise the construction for the same subset without $n + 1$ using the induction hypothesis, for the vertices 1 to n. It's easy to see that this is not f expanding only for $f = r$.

Now we do the case $r = n + 1$. For this case, we do this construction: k on the left is adjacent to all the vertices i on the right such that $i \leq k$, and $n + 1$ is joined to $\{1, 2, \ldots, n\}$. For any subset of the vertices on the left of size $g \leq n$, there is at least one vertex whose index is at least g, or there is $n + 1$ in it, so the size of $R(S)$ is $\geq g$, and thus $|R(S)| \geq |S|$. However since the set of neighbours of all the vertices on the left is of size n, the graph this construction leads to is not $n + 1$-nice.

This implies, by induction, that for every n, there for each $r \in \{1, 2, \ldots, n\}$, there exists a bipartite graph (with 2 n-sized partitions) that is not r-nice but is s-nice for all $s \neq r$, $r, s \in \{1, 2, \ldots, n\}$.

2.3. Marking scheme

1. Only mentioning the correct answer .... **(0 points)**
2. Showing that it suffices to prove that for every $r \in \{1, 2, \ldots, n\}$ there is a distribution that is not r-nice but is s-nice for all $s \in \{1, 2, \ldots, n\} \setminus \{r\}$ to solve the problem .... **(1 point)**
3. For every $r \in \{1, 2, \ldots, n\}$, constructing a distribution that is that is not r-nice but is s-nice for all $s \in \{1, 2, \ldots, n\} \setminus \{r\}$ .... **(6 points)**

   (a) constructing such a distribution for some particular value of r .... **(1 point)**

4. Claiming that a construction works without proving it (when it is indeed easy to see) .... **(-0 points)**

3. Problem 3

3.1. Problem

We call a set of integers *special* if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \ldots, 4n\}$ cannot be partitioned into n disjoint special sets.

Proposed by Mohsen Jamali, Iran

3.2. Solutions

We assume, for the sake of contradiction, that we can partition the set $\{1, 2, \ldots, 4n\}$ into n disjoint special sets $S_1, S_2, \ldots, S_n$. Every special set contains at most 2 even integers since it can be partitioned so that $ab - cd = 1$. But there're exactly $2n$ even and $2n$ odd numbers, thus each S_i must contain exactly 2 even and 2 odd integers, and the numbers with same parity are paired to each other. Let's call $S_i = \{a_i, b_i, c_i, d_i\}$ with a_i, b_i even, c_i, d_i odd, and $a_i b_i - c_i d_i = \pm 1$.

Note that $\bigcup_{i=1}^{n} \{a_i, b_i\} = \{2, 4, \ldots, 4n\}$ and $\bigcup_{i=1}^{n} \{c_i, d_i\} = \{1, 3, \ldots, 4n-1\}$.

Next, we provide two different solutions that lead to the contradiction of our assumption.

3.2.1. Solution 1 (Morteza Saghafian)

Observe that $a_i b_i = c_i d_i \pm 1 \le c_i d_i + 1 < (c_i + 1)(d_i + 1)$. Multiplying these equations for $i = 1, 2, \ldots, n$ yields

$$\begin{aligned} 2 \cdot 4 \cdot \ldots \cdot 4n &= \prod_{i=1}^{n} a_i b_i \\ &< \prod_{i=1}^{n} (c_i + 1)(d_i + 1) \\ &= 2 \cdot 4 \cdot \ldots \cdot 4n, \end{aligned}$$

which is impossible. This contradiction completes the proof.

3.2.2. Solution 2 (Natanon Therdpraisan)

Observe that $a_i b_i = c_i d_i \pm 1 \le c_i d_i + 1$. Multiplying all equations gives

$$2 \cdot 4 \cdot \ldots \cdot 4n = \prod_{i=1}^{n} a_i b_i \le \prod_{i=1}^{n} (c_i d_i + 1).$$

Consider any positive real numbers $x_1 < x_2 < x_3 < x_4$. By rearrangement inequality, we know that

$$(x_{j_1} x_{j_2} + 1)(x_{j_3} x_{j_4} + 1) \le (x_1 x_2 + 1)(x_3 x_4 + 1)$$

for any permutation (j_1, j_2, j_3, j_4) of $(1, 2, 3, 4)$.

Hence,

$$\begin{aligned} \prod_{i=1}^{n} (c_i d_i + 1) &\le \prod_{t=1}^{n} ((4t-3)(4t-1) + 1) \\ &< \prod_{t=1}^{n} (4t-2)(4t) \end{aligned}$$

$$= 2 \cdot 4 \cdot \ldots \cdot 4n,$$

which is clearly a contradiction and the proof completes.

3.2.3. Remark

The above solutions all approach by putting ± 1 on the odd side, multiplying all equations together, and then comparing the product of all even pairs and the product of all odd pairs (with ± 1 added to each pair). However, the solution can still be done similarly if ± 1 is put on the even side, i.e.

for solution 1, we use $c_i d_i = a_i b_i \pm 1 \geq a_i b_i - 1 > (a_i - 1)(b_i - 1)$;

for solution 2, rearrangement inequality implies $(x_{j_1} x_{j_2} - 1)(x_{j_3} x_{j_4} - 1) \geq (x_1 x_2 - 1)(x_3 x_4 - 1)$, and thus

$$\prod_{i=1}^{n} (a_i b_i \pm 1) \geq \prod_{t=1}^{n} ((4t-2)(4t) - 1) > \prod_{t=1}^{n} (4t-3)(4t-1).$$

Therefore, the marking scheme below should also apply equivalently if such approach happens.

3.3. Marking scheme

To our knowledge, we don't know any substantially different approach to this problem. If such approach happens, it should be judged as equivalently as possible. The marking scheme is divided into two additive parts. However, partial credits within each part are **not** additive.

1. Examples of non-rewarding observations: .... **(0 points)**
   (a) Casework specific value(s) of n that does not give insight to general case
   (b) Show that each special set has exactly one way of partition, e.g. if $ab - cd = 1$ then $|ac - bd|, |ad - bc| \neq 1$
   (c) Show that $gcd\ (a, c) = 1$ etc.
2. Showing that each special subset contains exactly 2 even and 2 odd integers, and numbers with same parity are paired to each other .... **(2 points)**
   (a) Prove that each special subset contains at most 2 even numbers .... **(1 point)**
   (b) Prove that each special subset contains at least 2 odd numbers .... **(1 point)**
   (c) Prove that each special subset contains exactly 2 even and 2 odd numbers .... **(1 point)**
3. Completing the solution .... **(5 points)**
   (a) Attempt to contradict the assumption by comparing the sizes of the product of all even pairs and that of all odd pairs, e.g. multiplying all n equations, or claiming that the product of all even pairs is still strictly more than the product of all odd pairs after plus 1 to each pair .... **(2 points)**
   (b) Show or mention (since it's easy to prove) that $xy \pm 1 < (x+1)(y+1)$ for any $x, y > 0$ .... **(2 points)**
   (c) Prove that the product of all odd pairs with 1 added to each pair maximizes when the numbers with more values are paired to each other .... **(2 points)**
   (d) Claim (c), without justification .... **(1 point)**
   (e) Show or mention that $xy \pm 1 \leq xy + 1$ for any real number x, y .... **(0 points)**

The correct solutions should be judged as 7 even if they're different from the above solutions. However, the following deductions could be applied:

4. The contestant has proven that each special subset contains 2 even and 2 odd integers, but use the fact that in each special subset each number must be paired only with one of same parity without mentioning .... **(-1 points)**
5. The contestant has any other non-trivial minor error .... **(-1 points)**

4. Problem 4

4.1. Problem

Prove that for all sufficiently large integers n, there exist n numbers $a_1, a_2, \ldots, a_n$ satisfying the following three conditions:

1. Each number a_i is equal to either -1, 0 or 1.

2. At least $2n/5$ of the numbers $a_1, a_2, \ldots, a_n$ are non-zero.

3. $a_1/1 + a_2/2 + \ldots + a_n/n = 0$.

Proposed by Navneel Singhal, India, Kyle Hess, USA, and Vincent Jugé, France

4.2. Solutions

4.2.1. Solution 1 (Vincent Jugé)

Let us say that a set $S \subseteq \{1, 2, \ldots, n\}$ is *nice* if there exists a function $f : S \mapsto \{-1, 1\}$ such that $\sum_{k \in S} f(k)/k = 0$; we say that f is a *witness* for S.

Thus, we prove below that, if n is large enough, there exists a nice set of size at least $2n/5$. Note a disjoint union of two nice sets is nice. Hence, the first goal would be to identify many disjoint nice sets. Aiming towards this goal, and due to the identity

$$1 - 1/2 - 1/3 - 1/6 = 0,$$

we first see that each set $S_k = \{k, 2k, 3k, 6k\}$, where $k \leq n/6$, is a nice set. However, we wish to have disjoint sets only, and therefore we will not consider all these sets at once.

Instead, let us check whether some integer m can belong to two such sets S_k and S_ℓ, with $n/12 < k < \ell \leq n/6$. Since

$$n/12 < k < \ell \leq n/6 < 2k < \min\{3k, 2\ell\} \leq \max\{3k, 2\ell\} < 3\ell \leq n/2 < 6k < 6\ell,$$

it follows that $m = 3k = 2\ell$.

This further implies that k is even, with $n/2 < k = 2\ell/3 \leq n/9$, and that ℓ is divisible by 3, with $n/8 < 3k/2 = \ell < n/6$.

Consequently, we can partition the set $\{k \in \mathbb{N} \colon n/12 < k \leq n/6\}$ into three subsets

$$\begin{aligned}
A &= \{k \in \mathbb{N} \colon n/12 < k \leq n/9 \text{ and } k \equiv 0 \pmod 2\}, \\
B &= \{k \in \mathbb{N} \colon n/8 < k \leq n/6 \text{ and } k \equiv 0 \pmod 3\} = \{3k/2 \colon k \in A\} \text{ and} \\
C &= \{k \in \mathbb{N} \colon n/12 < k \leq n/6\} \setminus (A \cup B),
\end{aligned}$$

such that both families $(S_k)_{k \in A \cup C}$ and $(S_k)_{k \in B \cup C}$ are families of pairwise disjoint nice sets. However, we cannot use simultaneously sets S_k and S_ℓ when $k \in A$ and $\ell \in B$.

Hence, we modify these sets as follows. Given some integer $k \in A$, let $\ell = 3k/2 \in B$. Due to the identity

$$\begin{aligned}
0 &= (1 - 1/2 - 1/3 - 1/6)/2 - (1 - 1/2 - 1/3 - 1/6)/3 - (1/6 - 2/12) \\
&= 1/2 - 1/3 - 1/4 - 1/6 + 1/9 + 1/12 + 1/18,
\end{aligned}$$

and although the intersection $S_k \cap S_\ell$ is non-empty, the set $S_k \cup S_\ell$ is still nice. Consequently, the sets $(S_k)_{k \in C} \cup (S_k \cup S_{3k/2})_{k \in A}$ form a family of pairwise disjoint nice sets. In particular, their union $\mathcal{S} = \bigcup_{n/12 < k \leq n/6} S_k$ is also nice, and its cardinality is $|\mathcal{S}| = 7|A| + 4|C| = 23n/72 - \mathcal{O}(1)$.

Finally, let $\mathcal{T}$ be a maximal nice set (for the inclusion) containing $\mathcal{S}$. Assume that some integer $k \leq n/12$ does *not* belong to $\mathcal{T}$. Let ℓ be the largest such integer, and let $f : \mathcal{T} \mapsto \{-1, 1\}$ be a witness for $\mathcal{T}$. By maximality of ℓ, we know that $2\ell \in \mathcal{T}$. Then, let $\mathcal{T}' = \mathcal{T} \cup \{\ell\}$, and let $f' : \mathcal{T}' \mapsto \{-1, 1\}$ be defined by $f'(\ell) = f(2\ell)$, $f'(2\ell) = -f(2\ell)$, and $f'(k) = f(k)$ otherwise. One checks easily that f' is a witness for $\mathcal{T}'$, contradicting the maximality of $\mathcal{T}$.

We conclude that each integer $k \leq n/12$ also belongs to $\mathcal{T}$, and thus that $\mathcal{T}$ is a nice set of size $|\mathcal{T}| \geq |\mathcal{S}| + n/12 + \mathcal{O}(1) = 29n/72 - \mathcal{O}(1)$. Since $29/72 > 2/5$, this completes the proof.

4.2.2. Solution 2 (Navneel Singhal)

We give an explicit construction of a nice tuple with $\frac{2}{5}$ replaced by $\frac{1}{3} - \varepsilon$, showing that the number of non-zero elements in such an $n-$tuple can be made $\frac{n}{3} - \mathcal{O}(\log^2 n)$. Let $r = \left\lfloor \frac{N}{6} \right\rfloor$. Let $v_p(n)$ be the highest e such that $p^e|n$ for a prime p. Let

$$a_k = \begin{cases} 1 & \text{if } v_2(k) \text{ and } v_3(k) \text{ are both even with } k \leq r \\ -1 & \text{if } v_2(k) \text{ is odd and } v_3(k) \text{ is even with } k \leq 2r \\ -1 & \text{if } v_2(k) \text{ is even and } v_3(k) \text{ is odd with } k \leq 3r \\ -1 & \text{if } v_2(k) \text{ and } v_3(k) \text{ are both odd with } k \leq 6r \\ 0 & \text{otherwise} \end{cases}$$

Firstly we show that this construction works. Consider the identity $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{6} = 0$. Consider integers m not exceeding r, of the form $2^{2x}3^{2y}a$, where $\gcd(a, 6) = 1$. Suppose the set of such m is $\mathbb{M}$. We claim that the set of terms of the sum $\sum_{k=1}^{n} \frac{a_k}{k}$ can be decomposed into 4−tuples of pairs which are of the form $\left(\frac{1}{m}, -\frac{1}{2m}, -\frac{1}{3m}, -\frac{1}{6m}\right)$ where m is of the aforementioned form, and this will show that $(a_1, \cdots, a_n)$ is indeed a nice tuple, because the sum of the elements in this tuple is 0, and summing this over all possible m of the aforementioned form and rearranging the terms we get our old summation.

Suppose $\mathbb{S}$ is the set $\mathbb{M} \cup 2\mathbb{M} \cup 3\mathbb{M} \cup 6\mathbb{M}$. Note that all the sets $2^a3^b\mathbb{M}$, $a, b \in \{0, 1\}$ are disjoint because by construction of the set $\mathbb{M}$, $2^a3^b\mathbb{M}$ is precisely the set of positive integers z such that $v_2(z) \equiv a \pmod 2$ and $v_3(z) \equiv b \pmod 2$ with $z \leq 2^a3^br$. Thus, $\mathbb{S}$ is the set of all k such that a_k is non zero in our construction. However this is also the set of all elements representable in the form 2^a3^bm where m is in $\mathbb{M}$ and $a, b \in \{0, 1\}$. Thus the set of absolute values of the non-zero a_k's is precisely the union of sets $\left\{\frac{1}{m}, \frac{1}{2m}, \frac{1}{3m}, \frac{1}{6m}\right\}$.

Now note that the sign of a_k is -1 if any of $v_2(k)$ or $v_3(k)$ is odd and $+1$ otherwise. The sign of any element $\frac{1}{w}$ in $\left(\frac{1}{m}, -\frac{1}{2m}, -\frac{1}{3m}, -\frac{1}{6m}\right)$ is -1 if any of $v_2(w)$ or $v_3(w)$ is odd and $+1$ otherwise. This implies that the set of the non-zero a_k's is precisely the union of the disjoint sets $\left\{\frac{1}{m}, -\frac{1}{2m}, -\frac{1}{3m}, -\frac{1}{6m}\right\}$, and thus we have shown that our construction satisfies the given conditions.

Clearly, the value of our constructed tuple is 4 times the number of elements in $\mathbb{M}$. We proceed to count the number of elements in $\mathbb{M}$.

For every pair (x, y), the number of integers w which have $v_2(w) = 2x$ and $v_3(w) = 2y$ is $\left\lfloor \frac{1}{3} \cdot \frac{r}{2^{2x}3^{2y}} \right\rfloor \geq \frac{1}{3} \cdot \frac{r}{2^{2x}3^{2y}} - 1$. We sum this over all (x, y) such that $\frac{r}{2^{2x}3^{2y}} \geq 1$ and get a lower bound on the number of terms. Suppose s is the number of solutions to $\frac{r}{2^{2x}3^{2y}} \geq 1$. Then we have

$$\begin{aligned} |\mathbb{M}| &\geq \sum_{\frac{r}{2^{2x}3^{2y}} \geq 1} \left(\frac{1}{3} \cdot \frac{r}{2^{2x}3^{2y}} - 1\right) \\ &= \frac{r}{3} \cdot \left(\sum_{\frac{r}{2^{2x}3^{2y}} \geq 1} \frac{1}{2^{2x}3^{2y}}\right) - s \\ &= \frac{r}{3} \cdot \left(\sum_{y=0}^{\left\lfloor \frac{\log_3 r}{2} \right\rfloor} \frac{1}{3^{2y}} \cdot \left(\sum_{x=0}^{\left\lfloor \frac{\log_2 \frac{r}{3^{2y}}}{2} \right\rfloor} \frac{1}{2^{2x}}\right)\right) - s \end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{3} \cdot \left(\sum_{y=0}^{\left\lfloor \frac{\log_3 r}{2} \right\rfloor} \frac{1}{3^{2y}} \cdot \frac{4}{3} \cdot \left(1 - \frac{1}{2^{2+2\left\lfloor \frac{\log_2 \frac{r}{3^{2y}}}{2} \right\rfloor}} \right) \right) - s \\
&> \frac{4r}{9} \cdot \left(\sum_{y=0}^{\left\lfloor \frac{\log_3 r}{2} \right\rfloor} \frac{1}{3^{2y}} \cdot \left(1 - \frac{1}{2^{\log_2 \frac{r}{3^{2y}}}} \right) \right) - s \\
&= \frac{4r}{9} \cdot \left(\sum_{y=0}^{\left\lfloor \frac{\log_3 r}{2} \right\rfloor} \left(\frac{1}{3^{2y}} - \frac{1}{r} \right) \right) - s \\
&= \frac{4r}{9} \cdot \left(\sum_{y=0}^{\left\lfloor \frac{\log_3 r}{2} \right\rfloor} \frac{1}{3^{2y}} \right) - s - \frac{4}{9} \cdot \left\lfloor \frac{\log_3 r}{2} \right\rfloor \\
&= \frac{4r}{9} \cdot \frac{9}{8} \left(1 - \frac{1}{3^{2+2\left\lfloor \frac{\log_3 r}{2} \right\rfloor}} \right) - s - \frac{4}{9} \cdot \left\lfloor \frac{\log_3 r}{2} \right\rfloor \\
&> \frac{r}{2} - \frac{1}{2} - s - \frac{4}{9} \cdot \left\lfloor \frac{\log_3 r}{2} \right\rfloor
\end{aligned}
$$

Now we compute s. $\frac{r}{2^{2x}3^{2y}} \geq 1$ is equivalent to $2x \log 2 + 2y \log 3 \leq \log r$. So we have $0 \leq 2x \leq \log_2 r$ and $0 \leq 2y \leq \log_3 r$. Since the solution lies inside a rectange, s doesn't exceed $\frac{1}{4} \log_2 r \log_3 r$. So we have $4|\mathbb{M}| > 2r - 2 - \frac{8}{9} \log_3 r - \log_2 r \log_3 r$, and we are done.

4.2.3. Solution 3 (Pitchayut Saengrungkongka)

We call a subset of $\{1, 2, \ldots, n\}$ nice if and only if we can assign signs to each element so that the sum of reciprocals of each element is zero. We aim to show that there exists a nice set of size $\frac{2}{5}n$. First, we begin with the following.

Claim. Suppose that S is a maximal nice set, then $2k \in S \implies k \in S$.

Proof. Trivial. Assume for the contradiction and do $\frac{1}{2k} \to \frac{1}{k} - \frac{1}{2k}$ to get the better set. □

Let $I = \left[\frac{n}{12}, \frac{n}{6}\right]$. For any number $k \in I$, we define $S_k = \{2k, 3k, 6k\}$. In light of the identity $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$, we aim to get much of the S_k's as possible. However, there are some overlaps that we must take care of.

We construct a graph G by having all numbers in I as vertices and draw edges from a to b if and only if $S_a \cap S_b \neq \emptyset$. We have the following claim.

Claim. For any $a, b \in I$, we have

- $|S_a \cap S_b| \leq 1$
- $|S_a \cap S_b| = 1$ if and only if $3a = 2b$ or $3b = 2a$.
- Graph G consists of only isolated vertices and matchings.

Proof. The first two can be checked through a simple calculation. For the third one, we note that if k is has degree two, then both $\frac{2}{3}k$ and $\frac{3}{2}k$ must be in I, which is impossible since $\frac{9}{4} > 2$. □

Now for any isolated vertices k, we add $\{2k, 3k, 6k\}$ to S. For any edge $2k \to 3k$, we add $\{4k, 9k, 12k, 18k\}$ to S in light of the identity $\frac{1}{4} = \frac{1}{9} + \frac{1}{12} + \frac{1}{18}$. Observe that this won't clash as $\{4k, 9k, 12k, 18k\} \subset S_{2k} \cup S_{3k}$.

Now we use the first claim to adjoin more elements. Right now, S we consider each element $k \in S$ based on residues modulo 6.

- If $k \equiv 0 \pmod 6$, then any $k < n$ is in S. This is because we can keep multiplying by 2 until we reach $\left[\frac{n}{2}, n\right]$.
- If $k \equiv 3 \pmod 6$, then any $k < \frac{n}{2}$ is in S. This is because $2k$ meets the above case.
- If $k \equiv 2, 4 \pmod 6$, then we claim that any $k < \frac{n}{3}$ is in S. To see this, note that $\{2^t \cdot k, 1.5 \cdot 2^t k, 3 \cdot 2^t \cdot k\}$ is one of those S_a's where t is picked so that $\frac{n}{2} < 3 \cdot 2^t \cdot k < n$. Since $2^t \cdot k$ is not a multiple of 3, it can't be clashed with other sets so $2^t \cdot k \in S \implies k \in S$.
- If $k \equiv 1, 5 \pmod 6$, then any $k < \frac{n}{6}$ is in S. This is because $2k$ meets the above case.

It's easy to compute the density of each case to get $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{9}$, and $\frac{1}{18}$. Thus we get the bound $\left(\frac{5}{12} - \epsilon\right) n$ for any $\epsilon > 0$.

4.3. Preliminary notes on grading this problem

The bound $2n/5$ is not sharp. The problem is also true if the bound $2n/5$ is replaced by cn where $0 < c < 1$. The following list gives the number of points awarded for a complete solution in function of the bound.

1. $o(n)$ non-zero terms .... **(0 points)**
2. cn non-zero terms for $c < 2/9$ and c cannot be arbitrarily close to $2/9$ .... **(1 point)**
3. cn non-zero terms for values of c arbitrarily close to (but smaller than) $2/9$ .... **(2 points)**
4. cn non-zero terms for $2/9 \leq c < 1/3$ and c cannot be arbitrarily close to $1/3$ .... **(2 points)**
5. cn non-zero terms for values of c arbitrarily close to (but smaller than) $1/3$ .... **(3 points)**
6. cn non-zero terms for $1/3 \leq c < 3/8$ and c cannot be arbitrarily close to $3/8$ .... **(4 points)**
7. cn non-zero terms for values of c arbitrarily close to (but smaller than) $3/8$ .... **(5 points)**
8. cn non-zero terms for some $3/8 \leq c < 2/5$, even for such c arbitrarily close to $2/5$ .... **(5 points)**
9. cn non-zero terms for some $c \geq 2/5$ .... **(7 points)**

4.4. Marking scheme

We start by a couple of ideas that are worth 0 points:

1. Writing down a relation of the type $1/k - 1/(k+1) - 1/k(k+1) = 0$ (for a given value of k or in general for all k).
2. Stating or proving that one can assemble disjoint nice subsets to build larger nice subsets.

Partial credits are distributed as follows. The final grade is the maximum number of points between the ones achieved according to the following scheme and the ones obtained for a complete solution with a different bound. The first point is non-additive to the rest.

1. Considering sets of the type $\{2k, 3k, 6k\}$ or $\{k, 2k, 3k, 6k\}$, for generic values of k. **(1 point) (non-additive)**
2. Considering a **linear** number of such sets, which are either disjoint or whose intersections form regular patterns such as between $\{k, 2k, 3k, 6k\}$ and $\{2k, 4k, 6k, 12k\}$, or between $\{2k, 4k, 6k, 12k\}$ and $\{3k, 6k, 9k, 18k\}$). . **(2 points)**
3. Correctly gluing these (intersecting) sets to form larger nice sets. .... **(2 points)**
4. Obtaining a nice set of the right cardinality. .... **(1 point)**
5. Estimating the cardinality of the obtained nice set and concluding that it has the right lower bound. .... **(1 point)**

5. Problem 5

5.1. Problem

Let $\mathbb{Q}$ denote the set of rational numbers. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that, for all $x, y \in \mathbb{Q}$,

$$f(x)f(y+1) = f(xf(y)) + f(x).$$

Proposed by Nicolás López Funes and José Luis Narbona Valiente, Spain

5.2. Solutions

We denote the given functional equation by $\mathbf{E}(x, y)$. The functions $f : x \mapsto 0$, $f : x \mapsto 2$ and $f : x \mapsto x$ are solutions. We prove below that there are no other solutions.

5.2.1. Solution 1 (Vincent Jugé)

Let f be a solution other than the two functions $x \mapsto 0$ and $x \mapsto 2$. First, $\mathbf{E}(0, x-1)$ states that $f(0)(f(x) - 2) = 0$ for all $x \in \mathbb{Q}$. Since f is not equal to the function $x \mapsto 2$, it follows that $f(0) = 0$.

Then, since f is not constantly zero, there exists a real number z such that $f(z) \neq 0$. The equality $\mathbf{E}(z, 0)$ states that $f(z)(f(1) - 1) = 0$, i.e., that $f(1) = 1$.

We prove now, by induction, that $f(n) = n$ for all integers $n \geq 0$. Indeed, this is already the case for $n = 0$ and $n = 1$ and, if $f(n) = n$ for some integer $n \geq 0$, the equality $\mathbf{E}(1, n)$ states that

$$f(n+1) = f(1)f(n+1) = f(f(n)) + f(1) = n + 1.$$

Furthermore, if $x = \frac{p}{q}$ is a positive rational number, with p and q positive integers, the equality $\mathbf{E}(x, q)$ proves that $qf(x) = p$, i.e., that $f(x) = x$.

Now, let us focus on negative rational numbers, and let $a = f(-1)$. The equality $\mathbf{E}(1, -1)$ states that $f(a) = -1$, and thus that $a < 0$. Then, if r is a negative rational number, the equality $\mathbf{E}(-1, -r)$ also states that $f(r) = -ar$. It follows that $-1 = f(a) = -a^2$ and, since $a < 0$, that $a = -1$.

Hence, we conclude that $f(x) = x$ for all $x \in \mathbb{Q}$, which completes the proof.

5.2.2. Solution 1a (Jhefferson Lopez)

We demonstrate an alternative way of getting $f(0) = 0$ if f is not constantly 2. The equation $\mathbf{E}(0, -1)$ tells us that $f(0)^2 = 2f(0)$, which means that either $f(0) = 0$ or $f(0) = 2$. If $f(0) = 2$, then $\mathbf{E}(0, x-1)$ implies that $f(x) = 2$ for all $x \in \mathbb{Q}$, which contradicts our assumption. Therefore $f(0)$ must be 0.

5.2.3. Solution 2 (Jhefferson Lopez)

Just like in solution 1 we show that if f is a solution other than $x \mapsto 0$ and $x \mapsto 2$, then $f(x) = x$ for all rational numbers $x \geq 0$.

From the equality $\mathbf{E}(1, -1)$ we get that $f(f(-1)) = -1$. Using this, the equality $\mathbf{E}(1, f(-1))$ tells us that $f(1 + f(-1)) = 1 + f(-1)$. The equality $\mathbf{E}(x, f(-1))$ states that

$$f(x)f(1 + f(-1)) = f(xf(f(-1))) + f(x)$$

which then, using the two facts we just derived, implies that

$$f(x)f(-1) = f(-x). \tag{1}$$

Plugging $x = -1$ into (1), we get that $f(-1)^2 = f(1) = 1$. This leaves us with the two cases $f(-1) = 1$ or $f(-1) = -1$. If $f(-1) = 1$, then we have $f(f(-1)) = f(1) = 1$, which is a contradiction to $f(f(-1)) = -1$. If $f(-1) = -1$, then (1) tells us that $f(x) = -f(-x)$ for all $x \in \mathbb{Q}$. Since we already know $f(x) = x$ for all rational $x \geq 0$, this tells us that $f(x) = x$ for all $x \in \mathbb{Q}$, which completes the proof.

Remark. We can also first show $f(n) = n$ for all integers $n \geq 0$, then show $f(x) = -f(-x)$ as mentioned above, and then use $\mathbf{E}(\frac{p}{q}, q)$ with integers $p, q \neq 0$ to conclude that $f(x) = x$ for all $x \in \mathbb{Q}$.

5.3. Marking scheme

In every sub-enumeration, the mentioned partials are given, if the mentioned part of the solution has not been completed.

1. Discovering all the solutions (without any additional "solutions" that do not actually satisfy the functional equation), and mentioning they all work[2] .... **(1 point)**
2. Proving that there can not be any other solutions .... **(6 points)**
   (a) Showing that if f is not constant, then $f(0) = 0$ and $f(1) = 1$ .... **(1 point)**
   (b) Showing that if f is not constant, then $f(n) = n$ for all integers $n \geq 2$ .... **(1 point)**
   (c) Showing that if f is not constant, then $f(x) = x$ for all rational $x > 0$ .... **(1 point)**
   (d) Showing that if f is not constant, then $f(n) = -f(-n)$ for all integers n[3] .... **(2 points)**
   (e) Any linear combination of the above .... **(Sum of the corresponding points)**

[2] Something along the lines of "it's obvious that they are solutions" is enough.

[3] This point should also be awarded if the contestant finds a way to show $f(n) = n$ for all negative integers n without deriving $f(n) = -f(-n)$ explicitly.

6. Problem 6

6.1. Problem

Decide whether there exist infinitely many triples (a, b, c) of positive integers such that all prime factors of $a! + b! + c!$ are smaller than 2020.

Proposed by Pitchayut Saengrungkongka, Thailand

6.2. Solutions

The answer is negative; there are finitely many such triples.

Let p be a prime greater than 2020 (for example, $p = 2027$). Without loss of generality, let $a \le b \le c$. If $a \ge p$, then $p \mid a! + b! + c!$ so we are done. Hence let us assume that $a < p$.

Claim 1. $b < a + p < 2p$

Proof. Assume that $b \ge a + p$. Then note that $p!a! \mid b!$

$$\frac{a! + b! + c!}{a!} = 1 + \frac{b! \cdot k}{a!} = 1 + p! \cdot \text{integer}$$

thus any prime factor of $\frac{a!+b!+c!}{a!}$ is greater than p. Obviously there exists at least one hence a contradiction. □

Claim 2. $c < p! + (2p)! + p$

Proof. Let $N = p! + (2p)!$. Assume that $c \ge N + p$. Notice that $a! + b! \mid N!$ thus $p!(a! + b!) \mid c!$ which means

$$\frac{a! + b! + c!}{a! + b!} = 1 + p! \cdot \text{integer}$$

thus any prime factor of $\frac{a!+b!+c!}{a!+b!}$ is greater than p. Obviously at least one exists so contradiction. □

The two claims above give the bound for a, b, c. Hence there are finitely many tuples.

Remark 1. *The problem generalizes to $a_1! + a_2! + \ldots + a_n!$. The proof is basically just keep applying the same argument until you get the bounds of all a_i's, which is quite techical and not worth asking.*

6.3. Marking Scheme

To our knowledge, we don't know any substantially different approach to this problem. If such approach happens, it should be judged as equivalently as possible.

The marking scheme is divided into three additive parts. However, the partial credits within each part are **not additive**.

In what follows, assume that $a \le b \le c$.

1. Examples of non-rewarding observations: any subset of the following. .... **(0 points)**

   (a) Try to construct such triples.

   (b) Assume that $a \le b \le c$ (or similar variants).

   (c) Reduce the problem to showing that c is bounded by a constant.

   (d) List all primes smaller than 2020.

   (e) Correctly guess the answer.

   (f) Prove that for a **chosen** constant $k \in \mathbb{Z}$, there exists only finitely many positive integer a such that all prime factors of $a! + k$ are smaller than 2020.
   (g) Prove the smaller case (i.e. when 2020 is replaced to smaller number), that does not show the idea of how to do the general case.

2. Showing that a must be bounded by a constant .... **(1 point)**
3. Showing that b must be bounded by a constant .... **(3 points)**
   (a) Make a serious attempt of using the fact that each prime factor of an integer $n \equiv 1 \pmod{k!}$ must be greater than k, or its similar variants. .... **(1 point)**
4. Showing that c must be bounded by a constant .... **(3 points)**

In addition, points could be given for contestants who have proposed and solved the easier version to this problem. These points are **not additive** to each other and are **not additive** to the mainstream progress above.

1. Prove that for **any** constant $k \in \mathbb{Z}$, there exists only finitely many positive integer a such that all prime factors of $a! + k$ are smaller than 2020. .... **(1 point)**
2. Prove that for there exists only finitely many pairs (a, b) of positive integers such that all prime factors of $a! + b!$ are smaller than 2020. .... **(2 points)**
3. Prove that for a **chosen nonzero** constant k, there exists only finitely many pairs (a, b) of positive integers such that all prime factors of $a! + b! + k$ are smaller than 2020. .... **(5 points)**

The correct solution should be judged as 7 points. However, the following deductions could be applied.

1. The contestant has an incorrect bound which is caused by **only** miscomputation. .... **(-0 points)**
2. The contestant has incorrectly do the first part, by claiming the contradiction if $a \geq 2020$, where, in principle a could lie between 2021 to 2026 (inclusive). .... **(-1 points)**
3. The contestant has any other nontrivial minor error. .... **(-1 points)**

7. Problem 7

7.1. Problem

Each integer in $1, 2, 3, \ldots, 2020$ is coloured in such a way that, for all positive integers a and b such that $a+b \leq 2020$, the numbers a, b and $a+b$ are not coloured with three different colours. Determine the maximum number of colours that can be used.

Proposed by Massimiliano Foschi, Italy

7.2. Solutions

Answer. In general, when the set is substituted by $\{1, 2, \ldots, n\}$, the answer is $\lfloor \log_2 n \rfloor + 1$. In this case, the answer is 11.

7.2.1. Example

A colouring which uses $\lfloor \log_2 n \rfloor + 1$ is as follows: colour with the i-th colour all the numbers m with $v_2(m) = i - 1$. As the maximum value $v_2(m)$ attains is $\lfloor \log_2 n \rfloor$, this colouring uses exactly $\lfloor \log_2 n \rfloor + 1$ colours. Note that, among a, b and $a + b$, at least two have the same 2-adic evaluation.

7.2.2. Bound

Let $k(n)$ be the maximum number of colours.

Approach 1 (Massimiliano Foschi)

We prove the following *lemma*: if $\{1, 2, \ldots, n\}$ cannot be coloured with more than x colours, then neither $\{1, 2, \ldots, 2n\}$ nor $\{1, 2, \ldots, 2n + 1\}$ can be coloured with more than $x + 1$ colours.

By the inductive hypothesis, the numbers in $\{2, 4, 6, \ldots, 2n\}$ are coloured with at most x colours, as they are simply the doubles of those in $\{1, 2, \ldots, n\}$. If the set were coloured with $x + 2$ colours or more, then there would be two odd numbers $d_1 < d_2$ in that set such that they are coloured differently and no even number is coloured with one of their colours. This yields to a contradiction by taking $a = d_1$ and $b = d_2 - d_1$.

Note that, if $n = x_m x_{m-1} \ldots x_0$ when written in binary, then $m = \lfloor \log_2 n \rfloor$. Now observe that

$$k(n) = k(x_m \ldots x_1 x_0) \leq k(x_m \ldots x_1) + 1 \leq k(x_m \ldots x_2) + 2 \leq \ldots \leq m + 1$$

Approach 2 (Massimiliano Foschi)

The lemma used in the solution in approach 1 can be proven in the following way:

note that if there are at least 2 colours present in $\{n + 1, \ldots, 2n\}$ (or $\{n + 1, \ldots, 2n + 1\}$ which are not present in $\{1, 2, \ldots, n\}$, then, by taking two numbers coloured with these two colours, their difference (which is $\leq n$) is coloured with a different colour, contradiction.

Thus $k(2n) \leq k(n) + 1$ and $k(2n + 1) \leq k(n) + 1$.

Approach 3 (Pitchayut Saengrungkongka)

Let $a_1 < a_2 < ... < a_s$ be the smallest number of each color.

Claim. $a_i \geq 2a_{i-1}$ for each i.

Proof. By minimality, a_i and $a_{i+1} - a_i$ both cannot have the same color as a_{i+1}, thus, they must have the same color. This is the contradiction if $a_{i+1} - a_i < a_i$.

Therefore $a_s \geq 2^{s-1}$, hence $s \leq \lfloor \log_2 n \rfloor + 1$.

7.3. Marking scheme

3 points will be given for the example (and the lower bound) and **4 points** will be given for the upper bound. Marks from different sections of the markscheme are additive, marks from the same section aren't. This markscheme is designed to be as general as possible, therefore if a partial solution does not follow the lines of one of these approaches, it is likely to be nevertheless rewarded appropriately.

If a corrector believes this is not the case for a particular solutions, they are recommended to debate it.

7.3.1. Example

Note. Throughout this part of the markscheme, for the example in this paper, the fact that it uses exactly 11 colours will be considered trivial.

1. Claiming that, $k(2020) = \lfloor \log_2 2020 \rfloor + 1$ or, equivalently, $k(2020) = 11$. .... **(1 point)**
2. Providing a valid example. .... **(1 point)**
3. Providing a valid example and proving that it uses 11 colours (or this fact is trivial). .... **(2 points)**
4. Providing a valid example and proving that it is valid (if the student has used the example in this paper, this can be done simply by citing the fact that $v_2(a)$, $v_2(b)$ and $v_2(a+b)$ are not all different as well-known. . **(2 points)**
5. The student satisfies both criteria for 2 points. .... **(3 points)**

7.3.2. Upper bound

1. Proving that $k(2n) \leq k(n) + 1$. .... **(1 point)**
2. Proving that $k(2n+1) \leq k(n) + 1$. .... **(3 points)**
3. Proving that, using the notation in approach 3, $a_i \geq 2a_{i-1}$. .... **(3 points)**
4. In general, proving a claim which easily yields the solution. .... **(3 points)**
5. Proving that $k(2020) \leq 11$. .... **(4 points)**

7.3.3. Deductions

1. The student miscalculates the answer. .... **(-1 points)**
2. The student miscalculates the answer but understands that it is $\lfloor \log_2 2020 \rfloor + 1$. .... **(-0 points)**

8. Problem 8

8.1. Problem

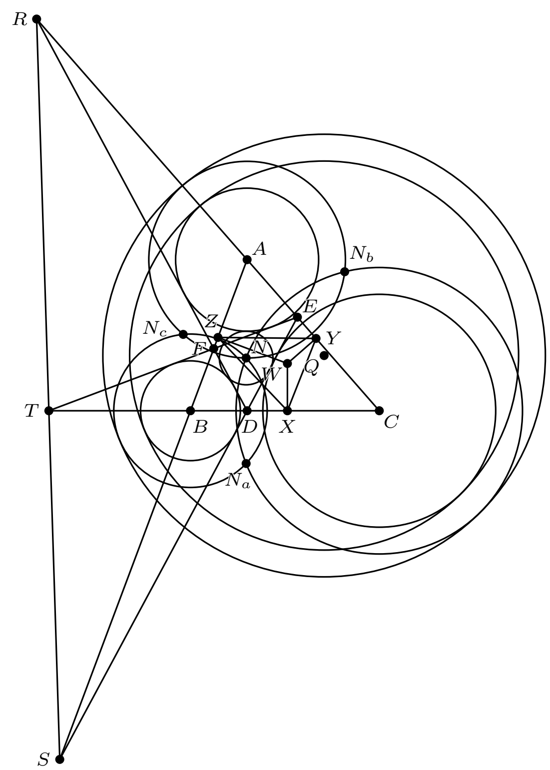
Let ABC be an acute scalene triangle, with the feet of A, B, C onto BC, CA, AB being D, E, F respectively. Suppose N is the nine-point centre of DEF, and W is a point inside ABC whose reflections over BC, CA, AB are W_a, W_b, W_c respectively. If N and I are the circumcentre and incentre of $W_aW_bW_c$ respectively, then prove that WI is parallel to the Euler line of ABC.

Note: If XYZ is a triangle with circumcentre O and orthocentre H, then the line OH is called the Euler line of XYZ and the midpoint of OH is called the nine-point centre of XYZ.

Proposed by Navneel Singhal, India and Massimiliano Foschi, Italy

8.2. Solutions

8.2.1. Solution 1 (Navneel Singhal)



Firstly note that since $AW_b = AW = AW_c$ and $NW_b = NW_c$, triangles AW_bN and AW_cN are congruent. Thus we have $\angle BAN = \angle WAC$. If N_a, N_b, N_c were the reflections of N in BC, CA, AB respectively, then by a similar argument, if the circumcenter of $N_aN_bN_c$ is W', then $\angle BAN = \angle W'AC$, so A, W, W' are collinear. Similarly, B, W, W' are collinear, and thus $W = W'$.

Since $N_bW = W_bN$, the circumradii of $N_aN_bN_c$ and $W_aW_bW_c$ are the same. Let the midpoint of WW_a be X (which is the foot of W onto BC). Define Y, Z, N_a', N_b', N_c' analogously. By a homothety at W with ratio $\frac{1}{2}$, the incenter of XYZ is mapped to the midpoint of W and the incenter of $W_aW_bW_c$. Note that $N_a'N_b'N_c'XYZ$ is cyclic due to this homothety and the analogous one at N, combined with the observation that the circumradii of $N_aN_bN_c$ and $W_aW_bW_c$ are the same. By the reflection in the center of this circle (which is the midpoint of NW by the homothety), if $N_a'N$ meets this circle again at N_A (and N_B, N_C are defined analogously), then the line joining W

and the incenter of XYZ is mapped to the line joining N and the incenter of $N_AN_BN_C$.

Consider the inversion at N with power $NN_A \cdot NN_a$, where the lengths are directed. Then N_BN_C is mapped to the circle passing through N, N_b, N_c, which is the circle centered at A, passing through N (say ω_A). The incircle of $N_AN_BN_C$ is mapped to a circle tangent to $\omega_A, \omega_B, \omega_C$. We show that this tangency is internal.

(Handling of configuration issues) Since W is inside ABC, N is also inside ABC (because of the angle relations). When ABC is an equilateral triangle, we know that the touching is internal. When we move A in the plane, and the touching changes from internal to external for at least one circle, since all our constructions are continuous, there must be a point where the transition happens, i.e., the common circle degenerates to a line or a point or one of $\omega_A, \omega_B, \omega_C$ degenerates to a line or a point (the last two of which are impossible as N is inside ABC and A, B and C are finite points). First we consider the case of the common circle degenerating to a line. By the *expansion* [4] which decreases the radius of the similarly oriented circles $\omega_A, \omega_B, \omega_C$ by the radius of the nine-point circle of DEF (or an analogous method to the second approach below), we have that there is a common tangent to all the excircles of DEF, such that all of them are on the same side of the line. Note that EF is a common tangent to the excircles, but it separates two pairs of excircles. Thus either the one of the excircles degenerates (in which case one of the angles of ABC becomes 90°, which is impossible), or the other direct tangent of the E, F-excircles is tangent to the D-excircle. Since the distance from A to EF is the same as the distance from A to this other tangent, which is in fact the reflection of EF over BC, it means that the A is on an angle bisector of this tangent and EF. Since A is not on BC, it must be on the other angle bisector, which is in fact perpendicular to BC. Thus EF passes through D, which is a contradiction, by Ceva's theorem, unless D is either of B or C, which is impossible because ABC is acute. Thus the common circle degenerates to a point instead, and since A, B, C are not collinear, $\omega_A, \omega_B, \omega_C$ are not coaxial, and thus the common circle must be N. This means that the incircle of $N_AN_BN_C$ has an infinite size, which is a contradiction. This shows that as long as we can move on a line through the position of A such that ABC is equilateral without contradicting the conditions, the tangency is internal. Now we claim that while moving A from the position such that ABC is an equilateral triangle to the original position, ABC stays acute and W never steps out of ABC (which would finish off the discussion of the configuration issues). For this, we need a preliminary claim: all angles of ABC are $> 30^\circ$. Note that the perpendicular bisector of EF contains the reflection of D over N, say O_D, which is the reflection of the circumcenter of DEF over EF. O_D would be on the other side of BC than A (or on BC) iff the midpoint of arc EDF is at least as close to EF as O_D, which is true iff $\angle BAC \le 30^\circ$, and we are done. Firstly we show that as A moves on the mentioned line, ABC remains acute. Since the angles at B and C change monotonically, and the final and the initial locations were such that these were acute in both cases, they always remain acute. It only remains to be shown that the angle at A remains acute. Suppose A_0 is the position of A such that A_0BC is an equilateral triangle and A, A_0 are on the same sides of BC. Note that since we have shown already that all the angles of ABC are at least 30°, the line AA_0 meets the circumcircle of A_0BC at a point between the midpoints of arcs A_0B and A_0C, say at T. Note that the point where BC subtends the maximum angle on a point on A_0T is where the circle through B, C and tangent to A_0T meets A_0T. Note that A_0T does not intersect the circle with diameter BC for any permissible T, so the largest angle is indeed acute, hence we are done by the bitonicity of the angle at A. Now suppose W steps onto or outside ABC. Then either W steps over a side or W coincides with a vertex. If it coincides with a vertex, N crosses the opposite side, which is impossible as the condition that all the angles of ABC are $> 30^\circ$ is maintained due to monotonicity/bitonicity of the angle at A (with a maximum in between in the second case) and the monotonicity of angles at vertices B and C. If it crosses a side, then N coincides with a vertex, which is impossible since the distance from a vertex to N is the sum of the corresponding exradius and half the circumradius of DEF by Feuerbach's theorem.

Call this circle ω and suppose Q is the center of ω. Then NQ passes through the incenter of $N_AN_BN_C$. Thus it suffices to show that NQ is parallel to the Euler line of ABC. Firstly, we show that Q is the center of a circle (Q) tangent internally to the excircles of DEF (which are centered at A, B, C respectively). We show it in two ways.

Approach 1. Consider the *expansion* which decreases the radius of the similarly oriented circles $\omega_A, \omega_B, \omega_C$ by the radius of the nine-point circle of DEF. Then N is mapped to the nine-point circle of DEF. Note that the image of ω_A is tangent to this circle since tangencies are preserved in expansions, so ω_A is mapped to the $D-$excircle of DEF by Feuerbach's theorem (as A is the $D-$excenter of DEF since $\angle BAC$ is acute). The image of ω is tangent

[4] For a reference, please refer to the following article: `http://jcgeometry.org/Articles/Volume2/JCG2013V2pp11-25.pdf`

to the images of $\omega_A, \omega_B, \omega_C$, which finishes the proof of our claim.

Approach 2. Consider the nine-point circle (N) of DEF. Since $QA + AN = r_\omega$ (where r_ω is the radius of ω), if r_D is the radius of the excircle of DEF and R' is the radius of the nine-point circle of DEF, then $r_\omega - R' = QA + AN - R' = QA + r_D$ (where the last equality follows from Feuerbach's theorem and the fact that A is the $D-$excenter of DEF since $\angle BAC$ is acute), and similarly $r_\omega - R' = QB + r_E$ and $r_\omega - R' = QC + r_F$. This implies the existence of a circle centered at Q which is tangent to all three excircles of DEF.

There are two ways to finish.

Approach 1. Suppose EF, FD, DE meet BC, CA, AB at T, S, R respectively. Since EF is directly tangent to the E and F-excircles of DEF, and T is on the line joining their centers, T is the exsimilicenter of these circles. By Menelaus' theorem (or Monge's theorem or power of a point), the line joining the tangency points of (N) with these circles passes through T, so since tangency is preserved under inversion, (N) is preserved under this inversion. Similarly, (Q) is preserved under this inversion. So the power of this inversion equals the power of T with respect to both these circles, and thus T is on the radical axis of $(N), (Q)$. Similarly, R and S are also on the mentioned radical axis. Thus $RST \perp NQ$. Now note that by the radical axis theorem on the circle with diameter BC, the circumcircle and the nine-point circle of ABC, T lies on the radical axis of the circumcircle and the nine-point circle of ABC. Similarly, R and S are also on the mentioned radical axis. Thus RST is perpendicular to the line joining the centers of these circles, which is precisely the Euler line of ABC. This gives us that NQ is parallel to the Euler line of ABC, as required.

Approach 2 (Massimiliano Foschi). Let D', E' and F' be the midpoints of EF, FD and DE, respectively. Let B' and C' the points where the E-excircle and the F-excircle of DEF are tangent to EF. Then $FB' = EC'$, which implies that D' lies on the radical axis of said excircles. As the line through their centers is the external angle bisector of $\widehat{EDF}$, their radical axis is the internal angle bisector of $\widehat{E'D'F'}$. Thus, the radical centers of the three excircles of DEF is I_1, the incenter of $D'E'F'$.

Consider the inversion with center I_1 which fixes the three excircles. Clearly it swaps Γ and Ω, hence I_1, N and Q are collinear. Let G be the barycenter of DEF. Consider the homothety, centered at G with factor -2. The image of I_1 is the incenter of DEF, which is the orthocenter of ABC. The image of N is the nine-point center of ABC. Therefore the image of I_1NQ is the Euler line of ABC, which yields the result.

8.2.2. Solution 2 (Pitchayut Saengrungkongka and Navneel Singhal)

Since ABC is acute, A, B, C are respectively the D, E, F-excenters of DEF. By Feuerbach's theorem, these excircles are all tangent to the nine-point circle of DEF. Let T_a, T_b, T_c be the contact points of the nine-point circle of DEF with the three excircles of DEF. Let X_a, X_b, X_c be the points where the tangents to the nine-point circle of DEF at $\{T_b, T_c\}$, $\{T_c, T_a\}$ and $\{T_a, T_b\}$ meet, respectively.

Since N is inside ABC, $\angle T_bNT_c < 180°$, and T_a and N are on the same side of T_bT_c (as T_a is on the segment AN and A and N are on the same side of T_bT_c which is because segments T_bT_c and AN do not intersect because T_b is on segment BN and T_c is on segment CN), so $\angle T_bT_aT_c$ is acute. Thus $T_aT_bT_c$ is acute, and the incenter of this triangle is N (An alternative fix is to note that since N is inside ABC, and T_a, T_b, T_c are on AN, BN, CN respectively, N is inside $T_aT_bT_c$, so $T_aT_bT_c$ is acute because of having its circumcenter lie inside it). By radical axes theorem on the E, F-excircles of DEF and the nine-point circle of DEF, note that X_a is on the radical axis of the E, F excircles.

Let D', E' and F' be the midpoints of EF, FD and DE, respectively. Let B' and C' the points where the E-excircle and the F-excircle of DEF are tangent to EF. Then $FB' = EC'$, which implies that D' lies on the radical axis of said excircles. As the line through their centers is the external angle bisector of $\widehat{EDF}$, their radical axis is the internal angle bisector of $\widehat{E'D'F'}$. Thus, the radical centers of the three excircles of DEF is I_1, the incenter of $D'E'F'$. Thus I_1X_a is perpendicular to BC.

Since $AW_b = AW = AW_c$ and $NW_b = NW_c$, AN is the perpendicular bisector of W_bW_c. Now note that $W_bW_c \perp AN \perp X_bX_c$, so we have $W_bW_c \parallel X_bX_c$. Similarly we have $W_aW_b \parallel X_aX_b$ and $W_cW_a \parallel X_cX_a$, so the triangles

$W_aW_bW_c$ and $X_aX_bX_c$ are homothetic. Since $WW_a \perp BC \perp I_1X_a$, we have $WW_a \parallel I_1X_a$. Similarly we have $WW_b \parallel I_1X_b$ and $WW_c \parallel I_1X_c$, so since $W_aW_bW_c$ and $X_aX_bX_c$ are homothetic, the quadrilaterals $W_aW_bW_cW$ and $X_aX_bX_cI_1$ are homothetic. Since N is the incenter of $X_aX_bX_c$ and I is the incenter of $W_aW_bW_c$, the pentagons $W_aW_bW_cWI$ and $X_aX_bX_cI_1N$ are homothetic because N and I are corresponding points in the homothetic triangles $W_aW_bW_c$ and $X_aX_bX_c$. Hence there exists a homothety mapping $W_aW_bW_cWI$ to $X_aX_bX_cI_1N$. This homothety sends WI to I_1N, and thus these lines are parallel. Hence, it suffices to show that I_1N is parallel to the Euler line of ABC.

Note that the Euler line of ABC is the line joining the nine-point center of ABC (which is the circumcenter of DEF) and the orthocenter of ABC (which, since ABC is acute, is the incenter of DEF). By a homothety at the centroid of DEF which takes DEF to its medial triangle, the circumcenter of DEF is taken to the circumcenter of its medial triangle (which is N), and the incenter of DEF is taken to the incenter of the medial triangle (which is I_1). Hence the Euler line of ABC is mapped to the line I_1N, so they are parallel, as needed.

8.2.3. Solution 3 (Pitchayut Saengrungkongka)

We present a highly motivated complex bash which takes inspiration from the previous solution. Let $\odot(DEF)$ be the unit circle. We use the standard set up where $D = a^2$, $E = b^2$ and $F = c^2$. It's well known that we can select the signs of a, b, c such that the coordinates of A, B, C are given by

$$A = ab + ac - bc,\ B = ba + bc - ac,\ C = ca + cb - ab,$$

since ABC is acute and A, B, C are the excenters of DEF. Since $N = \frac{a^2+b^2+c^2}{2}$, we can compute $N - A = \frac{(b+c-a)^2}{2}$. Therefore,

$$|N - A| = \frac{1}{2}|b + c - a|^2 = \frac{(b + c - a)(ab + ac - bc)}{2abc}$$

Now we will let point X, Y, Z be the complex numbers correspond to $\frac{N-A}{|N-A|}$, etc. Let O be the circumcenter of $\triangle DEF$. Notice that as N lies inside $\triangle ABC$, we get that O lies inside $\triangle XYZ$. Thus if we let $\triangle X_1Y_1Z_1$ be the tangential triangle of $\triangle XYZ$, we get that $\triangle X_1Y_1Z_1$ are homothetic with $\triangle XYZ$.

Now we will compute the coordinate of X_1. First, note that we have

$$X = \frac{abc(b + c - a)}{(ab + ac - bc)},\ \text{etc.}$$

Therefore we can compute with some effort that

$$\begin{aligned}
X_1 &= \frac{2YZ}{Y + Z} \\
&= \frac{2a^2b^2c^2(a + c - b)(a + b - c)}{abc[(a + c - b)(ac + bc - ab) + (a + b - c)(ab + bc - ac)]} \\
&= \frac{bc(a^2 + 2bc - b^2 - c^2)}{b^2 - bc + c^2}.
\end{aligned}$$

Thus we also have

$$\overline{X_1} = \frac{b^2c^2 + 2a^2bc - a^2b^2 - a^2c^2}{a^2bc(b^2 - bc + c^2)}$$

Now let $P = k(ab + bc + ca)$ be the point such that $X_1P \parallel AD$ where $k \in \mathbb{R}$. We want to show that k is symmetric within a, b, c as we will get the homothetic system $\triangle X_1Y_1Z_1 \cup O \cup P \sim \triangle W_aW_bW_c \cup I \cup W$. To do that, we note that

$$\frac{A - D}{\overline{A} - \overline{D}} = \frac{(a - b)(a - c)}{\overline{(a - b)(a - c)}} = a^2bc.$$

Hence we have

$$X_1 - k(ab + bc + ca) = a^2bc\overline{X_1} - k \cdot (a^2 + ab + ac)$$

Plugging everything in, we find

$$\begin{aligned} k(a^2-bc) &= \frac{b^2c^2+2a^2bc-a^2b^2-a^2c^2}{b^2-bc+c^2} - \frac{bc(a^2+2bc-b^2-c^2)}{b^2-bc+c^2} \\ &= \frac{(a^2-bc)(bc-b^2-c^2)}{b^2-bc+c^2} \\ &= -(a^2-bc) \end{aligned}$$

hence $k=-1$ so we are done by homothety.

8.2.4. Solution 4 (Francesco Sala and Pitchayut Saengrungkongka)

For convenience, we slightly change the notation by letting N be the nine-point center of $\triangle ABC$ instead and let M be the nine-point center of $\triangle DEF$ instead.

First, we note that $\odot(W_aW_bW_c)$ is the pedal circle of W w.r.t. $\triangle ABC$ dilated at W with ratio 2. Hence its center should be the isogonal conjugate of W w.r.t. $\triangle ABC$, thus M and W are isogonal conjugate w.r.t. $\triangle ABC$.

Claim. AN is the angle bisector of $\angle HAW$.

Proof. Let T be the Poncelet point of A, D, E, F. Since T lies on the pedal circle of A w.r.t. $\triangle DEF$, which is the D-excircle of $\triangle DEF$, it follows that T is the tangency of the nine-point circle and the D-excircle of $\triangle DEF$, hence $T \in AM$.

Let H_a be the foot from A to EF. From above, we deduce that $AH_a = AM$. Thus if N' is the nine-point center of $\triangle AEF$, then it follows that AN' bisects $\angle H_aAW$. Now notice that $\triangle AEF$ and $\triangle ABC$ are inversely similar thus $\{AN', AN\}$ are isogonal w.r.t. $\triangle ABC$. Thus by reflecting across the angle bisector of $\angle BAC$, we deduce the claim. □

Let O_a, O_b, O_c, and O be the circumcenters of $\triangle BMC$, $\triangle AMC$, $\triangle AMB$ and $\triangle ABC$ respectively. Clearly $\triangle W_aW_bW_c \cup W$ and $\triangle O_aO_bO_c \cup O$ are homothetic. Hence it suffices to show that the incenter I of $\triangle O_aO_bO_c$ lies on the Euler Line of $\triangle ABC$.

Claim. Let T_a, T_b, T_c be the reflection of the circumcenters of $\triangle BNC$,$\triangle CNA$, $\triangle ANB$ across BC, CA, AB respectively. Then M, T_a, T_b, T_c are concyclic with center I.

Proof. First, by directed angle chasing, we find that

$$\begin{aligned} \measuredangle BMC &= \measuredangle(BM,BC) + \measuredangle(CB,CM) \\ &= \measuredangle(BA,BW) + \measuredangle(CW,CA) \\ &= \measuredangle CWB + \measuredangle BAC \\ &= \measuredangle CWB + \measuredangle BHC + 2\measuredangle CAB \\ &= \measuredangle(CW,CH) + \measuredangle(BW,BH) + 2\measuredangle CAB \\ &= 2(\measuredangle(CN,CH) + \measuredangle(BN,BH) + \measuredangle BHC) \\ &= 2\measuredangle CNB \\ &= \measuredangle BT_aC \end{aligned}$$

Thus $T_a \in \odot(BMC)$. Therefore if $\triangle O'_aO'_bO'_c$ be the dilated image of $\triangle O_aO_bO_c$ under homothety $\mathcal{H}(M,2)$, it follows that $O'_a \in \odot(BMC)$ thus $O'_aT_a, O'_bT_b, O'_cT_c$ are internal bisectors of $\triangle O'_aO'_bO'_c$ (internal because T_a lies inside ABC so O'_a, T_a lie on different sides w.r.t. ABC because B, C, O'_a, T_a are concyclic) hence they must concur at I'. Thus $\angle MT_aI' = 90°$ which means that T_a, T_b, T_c lies on the circle with diameter MI', thus it's centered at I. □

Thus we will be done if we prove the following lemma (notice that N_a is the midpoint of NT_a).

Lemma. Let $\triangle ABC$ be a triangle with nine-point center N. Let N_a, N_b, N_c be the nine-point centers of $\triangle BNC$, $\triangle CNA$, $\triangle ANB$. Then the circumcenter of $\triangle N_aN_bN_c$ lies on the Euler Line of $\triangle ABC$.

Proof. Let O_a, O_b, O_c, O be the circumcenters of $\triangle BNC, \triangle CNB, \triangle ANB, \triangle ABC$ respectively. First, we show that $\triangle N_aN_bN_c \cup N \sim \triangle O_aO_bO_c \cup O$. Let $M_a, M_b, M_c, A', B', C'$ be midpoints of BC, CA, AB, AP, BP, CP respectively. Let V be the Poncelet point of $ABNC$. Notice that $NN_a \perp M_aV$ and $N_bN_c \perp A'V$ thus

$$\measuredangle N_bN_aN_c = \measuredangle C'VB' = \measuredangle C'M_aB' = \measuredangle BNC = \measuredangle O_cO_aO_b$$
$$\measuredangle N_bNN_c = \measuredangle M_bVM_c = \measuredangle M_bM_aM_c = \measuredangle BAC = \measuredangle O_cOO_b$$

which gives the desired similarity. Now, let K be the Kosnita point of $\triangle ABC$ and M be the midpoint of ABC. Let $\triangle P_aP_bP_c$ be the pedal triangle of N w.r.t. $\triangle ABC$, which has the circumenter M. We claim that M maps to N under this similarity. To prove that, note that $V \in \odot(P_aP_bP_c)$ (by property of Poncelet point and angle chasing) thus

$$\measuredangle MN_bN_c = \measuredangle P_bVA' = \measuredangle P_bM_bA' = \measuredangle ACN = \measuredangle O_cO_bN$$

which gives $\triangle N_aN_bN_c \cup N \cup M \cup \triangle O_aO_bO_c \cup O \cup N$. In particular, if X, T is the circumcenter of $\triangle N_aN_bN_c$ and $\triangle O_aO_bO_c$, we get that $\measuredangle XNM = -\measuredangle TON = \measuredangle NOT$. It suffices to show that $OT \parallel NK$.

To that end, let $\triangle K_aK_bK_c$ be the pedal triangle of K w.r.t. $\triangle ABC$. Notice that $\triangle K_aK_bK_c \cup M \cup K$ and $\triangle O_aO_bO_c \cup T \cup O$ are homothetic hence $OT \parallel MK$ as desired. $\square$

We have proven the last lemma. So we are done.

8.3. Marking Scheme

8.3.1. Preliminary remarks

1. All complete solutions should be awarded a full 7 points, regardless of whether they fit in the marking scheme or not.
2. Any solution (either computational or synthetic) which does not handle configuration issues should get a 1 point deduction.
3. Any computational solution that has minor computational errors should have a 1 point deduction.
4. Partial synthetic solutions should be judged as equivalently as possible.
5. Partial solutions using computational techniques should be given partial credit only if there is a synthetic interpretation mentioned in them, which could lead to a complete solution.
6. Partial credits across different approaches are **not** additive. If a student has a partial solution that can be graded via different marking schemes, the one which leads to the highest score should be followed.
7. Partial credits within one solution are additive.
8. Non-trivial but minor errors in a complete solution usually lead to a deduction of 1 point.

8.3.2. Solution 1 (Navneel Singhal)

1. Reaching to the point where $N_AN_BN_C$ is considered .... **(1 point)**
2. Inversion at N sending the incircle to the bigger tangent circle .... **(2 points)**
3. Showing that NQ is parallel to the Euler line .... **(3 points)**
4. Handling configuration issues .... **(1 point)**

8.3.3. Solution 2 (Pitchayut Saengrungkongka and Navneel Singhal)

1. Considering the points X_a etc .... **(2 points)**
2. Showing that X_aI_1 is the radical axis of the E, F-excircles .... **(2 points)**
3. Showing the existence of a homothety between the pentagons (deduct this point if configuration issues are not handled in an essentially complete solution) .... **(1 point)**
4. Using the homothety that maps DEF to its medial triangle .... **(2 points)**

8.3.4. Solution 3 (Pitchayut Saengrungkongka)

1. Constructing X, Y, Z .... **(2 points)**
2. Completing the complex bash correctly .... **(5 points)**

8.3.5. Solution 4 (Francesco Sala and Pitchayut Saengrungkongka)

1. Showing the first claim .... **(3 points)**
2. Showing the second claim .... **(2 points)**
   (a) Showing that $T_a \in \odot(BMC)$ .... **(1 point)**
3. Showing the final lemma .... **(2 points)**