GQMO

ADVANCED LEVEL

DAY 1

Each problem is worth 7 points.

These problems are to be kept confidential till Monday, 18th May 2020, 1200 hours (GMT).

Problem 1

Let ABC be a triangle with incentre I. The incircle of the triangle ABC touches the sides AC and AB at points E and F, respectively. Let ℓ_B and ℓ_C be the tangents to the circumcircle of BIC at B and C, respectively. Show that there is a circle tangent to EF, ℓ_B and ℓ_C with centre on the line BC.

Problem 2

Geoff has an infinite stock of sweets, which come in n flavours. He arbitrarily distributes some of the sweets amongst n children (a child can get sweets of any subset of all flavours, including the empty set). Call a distribution of sweets k-*nice* if every group of k children together has sweets in at least k flavours. Find all subsets S of $\{1, 2, \ldots, n\}$ such that if a distribution of sweets is s-nice for all $s \in S$, then it is s-nice for all $s \in \{1, 2, \ldots, n\}$.

Problem 3

We call a set of integers *special* if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \ldots, 4n\}$ cannot be partitioned into n disjoint special sets.

Problem 4

Prove that, for all sufficiently large integers n, there exist n numbers $a_1, a_2, \ldots, a_n$ satisfying the following three conditions:

- Each number a_i is equal to either -1, 0 or 1.
- At least $2n/5$ of the numbers $a_1, a_2, \ldots, a_n$ are non-zero.
- The sum $a_1/1 + a_2/2 + \ldots + a_n/n$ is 0.

Note: Results with $2/5$ replaced by a constant c will be awarded points depending on the value of c.

Advanced level

Day 2

Each problem is worth 7 points.

These problems are to be kept confidential till Monday, 18th May 2020, 1200 hours (GMT).

Problem 5

Let $\mathbb{Q}$ denote the set of rational numbers. Determine all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that, for all $x, y \in \mathbb{Q}$,

$$f(x)f(y+1) = f(xf(y)) + f(x)$$

Problem 6

Decide whether there exist infinitely many triples (a, b, c) of positive integers such that all prime factors of $a! + b! + c!$ are smaller than 2020.

Problem 7

Each integer in $\{1, 2, 3, \ldots, 2020\}$ is coloured in such a way that, for all positive integers a and b such that $a + b \leq 2020$, the numbers a, b and $a + b$ are not coloured with three different colours. Determine the maximum number of colours that can be used.

Problem 8

Let ABC be an acute scalene triangle, with the feet of A, B, C onto BC, CA, AB being D, E, F respectively. Let W be a point inside ABC whose reflections over BC, CA, AB are W_a, W_b, W_c respectively. Finally, let N and I be the circumcentre and incentre of $W_aW_bW_c$ respectively. Prove that, if N coincides with the nine-point centre of DEF, the line WI is parallel to the Euler line of ABC.

Note: If XYZ is a triangle with circumcentre O and orthocentre H, then the line OH is called the Euler line of XYZ and the midpoint of OH is called the nine-point centre of XYZ.